Special Hermitian metrics on Lie groups

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Impossible problem in complex geometry

Impossible problem. Classify all compact complex manifolds.

A complex manifold is (M, J), where

- M is an even-dimensional smooth manifold,
- $J : TM \to TM$ is an (integrable) complex structure.

We stratify the landscape using invariants.

In this talk: invariant = admitting "special" Hermitian metrics.

Kähler manifolds

A Hermitian metric on (M, J) is a Riemannian metric g on M satisfying

$$g(J\cdot, J\cdot) = g(\cdot, \cdot).$$

The fundamental 2-form of g is $\omega := g(J\cdot, \cdot)$.

We say that a Hermitian metric g is Kähler if $d\omega = 0$.

We say that a complex manifold (M, J) is Kähler if it admits a Kähler metric.

- Tori $\mathbb{C}^n/\Lambda$.
- Projective space $\mathbb{P}^n$.
- Any complex curve.
- Submanifolds of Kähler manifolds are Kähler,
  $\rightsquigarrow$ Projective complex manifolds are Kähler.

Balanced manifolds

A Hermitian metric g on (M^{2n}, J) is balanced if

$$d(\omega^{n-1}) = 0, \qquad \text{where } \omega^{n-1} = \underbrace{\omega \wedge \cdots \wedge \omega}_{n-1 \text{ times}}.$$

$$\text{Kähler} \Rightarrow \text{balanced:} \quad d\omega^{n-1} = (n-1) d\omega \wedge \omega^{n-2}.$$

If $d\omega^k = 0$ for some $k = 1, \ldots, n-2$, then $d\omega = 0$ (Popovici).

Example: Iwasawa manifold

$$\mathcal{H}_3(\mathbb{C}) = \left\{ \begin{pmatrix} 1 & z^1 & z^3 \\ 0 & 1 & z^2 \\ 0 & 0 & 1 \end{pmatrix} : z^k \in \mathbb{C} \right\}.$$

- $G = \mathcal{H}_3(\mathbb{C})$ has a complex structure J_{std} via $G \cong \mathbb{C}^3$.
- J_{std} is bi-invariant: left and right translations are biholomorphisms. In particular, J_{std} is left-invariant.
- $\Gamma = \mathcal{H}_3(\mathbb{Z}[i])$ is a cocompact lattice: $M = \Gamma \backslash G$ is compact.
- J_{std} descends to a complex structure J_{std} on $M = \Gamma \backslash G$.
- (M, J_{std}) admits no Kähler metrics.
- $\omega = i \sum_{k=1}^{3} \varepsilon^k \wedge \overline{\varepsilon^k}$ satisfies $d\omega^2 = 0$, where

$$\varepsilon^1 = dz^1, \qquad \varepsilon^2 = dz^2, \qquad \varepsilon^3 = dz^3 - z^1 dz^2.$$

ω is left-invariant, $\rightsquigarrow$ (M, J_{std}) is balanced.

Pluriclosed manifolds

A Hermitian metric g on (M, J) is pluriclosed if $\partial\overline{\partial}\omega = 0$.

$$\text{Kähler} \Rightarrow \text{pluriclosed:} \qquad d\omega = \underbrace{\partial\omega}_{\in\Lambda^{2,1}} + \underbrace{\overline{\partial}\omega}_{\in\Lambda^{1,2}}.$$

$$\partial\omega = 0 \quad \Leftrightarrow \quad \overline{\partial}\omega = 0 \quad \Leftrightarrow \quad \omega \text{ is Kähler.}$$

Theorem (Gauduchon). Every compact complex surface admits a pluriclosed metric.

Example. Hopf surfaces are pluriclosed but not Kähler.

$$(M, J) = (\mathbb{C}^2 \backslash \{0\})/\mathbb{Z}, \qquad k \cdot (z^1, z^2) = \frac{1}{2^k}(z^1, z^2).$$

$$M \cong_{\text{diffeo}} \mathbb{S}^1 \times \mathbb{S}^3 \quad \rightsquigarrow \quad b_1(M) = 1 \quad \rightsquigarrow \quad (M, J) \text{ not Kähler.}$$

The Fino–Vezzoni conjecture

If (M, J) is Kähler, then it is both balanced and pluriclosed.

Conjecture (Fino–Vezzoni): If a compact complex manifold is both balanced and pluriclosed, then it is Kähler.

Remark: If a metric ω is both balanced and pluriclosed, then ω is Kähler (Popovici).

The conjecture is obvious in complex dimensions 1 and 2.

Remark: Compactness cannot be dropped (Freibert–Swann).

Theorem (Fino–Vezzoni). The FV conjecture is true for $(\Gamma\backslash G, J)$, where G is nilpotent, Γ is a cocompact lattice and J is left-invariant.

Theorem (K). The FV conjecture is true for $(\Gamma\backslash G, J)$ where G is semisimple, Γ is a cocompact lattice and J is regular.

Left-invariant complex structures

Let G be a real Lie group, $\mathfrak{g}$ the Lie algebra of G.

Left-invariant J is encoded by $J : \mathfrak{g} \to \mathfrak{g}$.

$$J : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}} \text{ diagonalisable} \quad \rightsquigarrow \quad \mathfrak{g}^{\mathbb{C}} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1},$$

where $\mathfrak{g}^{1,0}$ and $\mathfrak{g}^{0,1}$ are the $+i$ and $-i$ eigenspaces, respectively.

$$J \text{ integrable} \quad \rightsquigarrow \quad \mathfrak{g}^{1,0} \text{ and } \mathfrak{g}^{0,1} \text{ are subalgebras.}$$

Tool 1. Left-invariant J on G is encoded by $\mathfrak{g}^{1,0}$.

Tool 2. Let Γ be a cocompact lattice of G, and $M = \Gamma \backslash G$. Fix a left-invariant J on G. TFAE:

- (G, J) has a left-invariant Kähler/balanced/pluriclosed metric.
- (M, J) has a Kähler/balanced/pluriclosed metric.

Complex Lie groups

Proposition. The FV conjecture is true for $(\Gamma\backslash G, J_{\mathrm{std}})$, where (G, J_{std}) is a complex Lie group, i.e. J_{std} is bi-invariant.

- The complexification of $\mathfrak{g}$ is $\mathfrak{g}^{\mathbb{C}} \cong \mathfrak{g} \oplus \overline{\mathfrak{g}}$,
- $J : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$, is given by $(X, Y) \mapsto (iX, -iY)$.
- $\mathfrak{g}^{1,0} = \mathfrak{g} \oplus 0$ and $\mathfrak{g}^{0,1} = 0 \oplus \overline{\mathfrak{g}}$ are ideals.

Let ω be any left-invariant Hermitian metric. Let E_k be an orthonormal basis for $\mathfrak{g}^{1,0}$ with dual basis ε^k.

$$\omega = i \sum_k \varepsilon^k \wedge \overline{\varepsilon^k}, \quad [E_j, E_k] = c_{jk}^\ell E_\ell, \quad d\varepsilon^\ell = -\sum_{j<k} c_{jk}^\ell \, \varepsilon^j \wedge \varepsilon^k.$$

- ω is balanced $\Leftrightarrow$ $\operatorname{tr}\operatorname{ad}(E_j) \equiv 0$, i.e. G is unimodular.
- ω is pluriclosed $\Leftrightarrow$ $[E_i, E_j] \equiv 0$, i.e. G is abelian.

Real semisimple Lie groups

Let G be a real semisimple Lie group, i.e. $\mathfrak{g}$ is a sum of real simple Lie algebras.

- Real simple Lie algebras are classified.
- Every semisimple G admits a cocompact lattice Γ.
- If J is left-invariant then $(\Gamma \backslash G, J)$ is not Kähler.

The Cartan involution of $\mathfrak{g}$ is the unique involutive automorphism $\theta : \mathfrak{g} \to \mathfrak{g}$ such that $-B(\cdot, \theta \cdot)$ is positive-definite.

We say G is inner if $\theta : \mathfrak{g} \to \mathfrak{g}$ is an inner automorphism.

- For compact G, $\theta = \mathrm{Id}_{\mathfrak{g}}$, so compact $\Rightarrow$ inner.
- If G is complex, then G is **not** inner (Knapp).

Regular complex structures

Fix a real semisimple Lie group G. A left-invariant J on G is called regular if there exists a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that

$$J \circ \mathrm{ad}(X) = \mathrm{ad}(X) \circ J \qquad \forall X \in \mathfrak{h}.$$

Remark. The right translations of $H \leq G$ with Lie algebra $\mathfrak{h}$ are J-biholomorphic.

- On compact G, Samelson complex structures are regular.
- On inner G, all left-invariant Js are regular (Snow).
- Snow classifies all regular J, these always exist on G^{2n}.
- Left-invariant Js are not classified in general.

Example. If (G, J_{std}) is a complex Lie group, then J_{std} is regular.

Main result

- On compact G^{2n}, left-invariant pluriclosed Js are classified by (Lauret–Montedoro), and these always exist. There are no balanced Js on G (FGV).
- On inner simple G^{2n}, (Giusti–Podesta) construct a left-invariant J such that $(\Gamma\backslash G, J)$ is balanced but not pluriclosed.

Theorem (K). For any semisimple G, cocompact lattice Γ, and regular J, $(\Gamma\backslash G, J)$ is exactly one of the following:

- Balanced.
- Pluriclosed.
- Neither balanced nor pluriclosed.

Example. There is a left-invariant J on $G = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ such that $(\Gamma\backslash G, J)$ is pluriclosed but not balanced.

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