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Isometries of Symmetric Spaces

Honours Thesis

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Abstract

In this thesis, we explicitly compute the full isometry groups of the ten infinite families of irreducible symmetric spaces of non-compact type. These spaces are in bijection with the (non-exceptional) real simple Lie algebras of non-compact type. We leverage this correspondence to construct explicit models for each family, either as spaces of positive-definite matrices or as open subsets of matrix vector spaces.

A well-known isomorphism identifies the isometry group of a symmetric space with the automorphism group of its Lie algebra. Building on the literature, we give a complete description of these automorphism groups. Finally, by translating via the isomorphism, we obtain explicit realisations of the isometry groups as transformation groups acting on the models.

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Contents

Chapter 1

Introduction

1.1 Overview

A central focus in modern geometry is the study of *Riemannian manifolds*. Roughly speaking, a Riemannian manifold is a metric space, where we are allowed to use tools from differential calculus. Familiar examples of Riemannian manifolds include the Euclidean plane $\mathbb{R}^2$ and the sphere $\mathbb{S}^2$.

The symmetry of a Riemannian manifold is captured by *isometries*: bijections which preserve distances. Some familiar examples of isometries are translations, rotations, and reflections of the plane, or rotations and reflections of the sphere.

The spaces $\mathbb{R}^2$ and $\mathbb{S}^2$ share a special property concerning their isometries: given any two points in the space, there is an isometry taking one point to the other. Riemannian manifolds with this property are called *homogeneous spaces*. A person standing inside a homogeneous space cannot tell where they are: the geometric information at every point is identical. There are exactly three 2-dimensional homogeneous spaces[1]: the Euclidean plane $\mathbb{R}^2$, the sphere $\mathbb{S}^2$, and the *hyperbolic plane* $\mathcal{H}^2$. The hyperbolic plane is more difficult to imagine than the other two, because it cannot be embedded inside Euclidean space $\mathbb{R}^3$ [dC16, §5-11, Hilbert's theorem]. Nevertheless, $\mathcal{H}^2$ can be realised as an open unit disk in the plane, endowed with a non-Euclidean distance function. In arbitrary dimensions, there is no hope of classifying homogeneous spaces[2].

The spaces $\mathbb{R}^2$, $\mathbb{S}^2$, and $\mathcal{H}^2$ share an even stronger property than homogeneity: given any two orthonormal bases (possibly at different points), there is an isometry sending one basis to the other. Riemannian manifolds with this property are called *frame-homogeneous spaces*. Frame-homogeneous spaces are classified in full generality. Indeed, the frame-homogeneous spaces are precisely the higher-dimensional analogs of $\mathbb{R}^2$, $\mathbb{S}^2$, and $\mathcal{H}^2$: they are the Euclidean spaces $\mathbb{R}^n$, the spheres $\mathbb{S}^n$, and the hyperbolic spaces $\mathcal{H}^n$ [Lee18, Lemma 8.33].

Symmetric spaces lie somewhere in between frame-homogeneous spaces and homogeneous spaces:

$$\Big\{\mathbb{R}^n,\, \mathbb{S}^n,\, \mathcal{H}^n\Big\} \subsetneq \Big\{\text{Symmetric spaces}\Big\} \subsetneq \Big\{\text{Homogeneous spaces}\Big\}.$$

A homogeneous space is called a *symmetric space* if at any point p, there is an isometry which fixes p and reverses the direction of all *geodesics* (straight lines) passing through p. This special isometry is called the *geodesic symmetry* at p. For example, the geodesic symmetry of $\mathbb{R}^2$ or $\mathbb{S}^2$ at a point is the rotation by 180 degrees around that point.

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[1] Unless stated otherwise, we assume our Riemannian manifolds are simply-connected.

[2] Homogeneous spaces are classified in dimension three: see [Pat96].

Symmetric spaces were discovered by Élie Cartan in 1926, near the end of his mathematical career [CC52]. His remarkable discovery was that symmetric spaces are in correspondence with simple real Lie algebras. Using this correspondence, Cartan classified all symmetric spaces: Cartan himself had classified all simple real Lie algebras in 1914. This correspondence simplifies computations on symmetric spaces. The geometric features of a symmetric space are completely determined by the Lie algebraic data, which is well-understood. Thanks to their computational tractability, symmetric spaces form an important testing ground for new ideas and conjectures in Riemannian geometry.

Beyond their roots in differential geometry and Lie theory, symmetric spaces permeate many other fields of mathematics, such as complex geometry (through Hermitian symmetric spaces [Hel78, Chapter VIII]), analytic number theory (through modular forms [BvdGHZ08]), algebraic geometry (via symmetric varieties [DCP83]), and even knot theory (through quandles [Tak16]).

The building blocks of symmetric spaces are the *irreducible* symmetric spaces: a symmetric space is called irreducible if it cannot be factored into a product of two symmetric spaces. For example, the sphere and hyperbolic plane are irreducible, but the Euclidean plane is not, because $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. Every symmetric space can be decomposed into a product of irreducible factors. Thus, the study of symmetric spaces reduces to the study of the irreducible ones. Ignoring the line $\mathbb{R}$, the irreducible symmetric spaces come in two types: compact and non-compact. Moreover, there is a duality between these two types. For example, the sphere is of compact type, the hyperbolic plane is of non-compact type, and there is a duality between the sphere and the hyperbolic plane.

The isometry group of a Riemannian manifold is a basic but fundamental invariant. The isometry groups of the frame-homogeneous spaces $\mathbb{R}^n$, $\mathbb{S}^n$ and $\mathcal{H}^n$ are explicitly known [Lee18, Problem 5-11]. Despite the extensive literature on the isometry groups of symmetric spaces (see §1.2), to the best of the author's knowledge, explicit descriptions of the full isometry groups of most symmetric spaces are still missing in the literature. The goal of this thesis is to remedy this in the non-compact case:

Goal. *Explicitly determine the full isometry groups of the irreducible symmetric spaces of non-compact type.*

1.2 Literature

Henceforth, the reader is assumed to be fluent in the languages of differential geometry and Lie theory. Results from [Hel78] give a bijective correspondence

$$\left\{ \begin{matrix} \text{Irreducible symmetric spaces} \\ \text{of non-compact type} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{Real simple Lie algebras} \\ \text{of non-compact type} \end{matrix} \right\}.$$

One direction of the correspondence is easy to describe: given an irreducible symmetric space of non-compact type (M, g), the corresponding Lie algebra is the Lie algebra of the isometry group, $\operatorname{Lie} \operatorname{Isom}(M, g)$.

The other direction is more involved, but also more useful to us: given a simple Lie algebra $\mathfrak{g}$ of non-compact type, the corresponding symmetric space can be constructed as follows. Let G denote the adjoint group of $\mathfrak{g}$ (i.e. the identity component of the automorphism group of $\mathfrak{g}$), and let K be a maximal compact subgroup of G. The corresponding symmetric space is $(G/K, g)$, where g is any G-invariant metric on G/K.

[Hel78, Chapter VI Exercise 7] tells us that the adjoint representation

$$\operatorname{Ad} : \operatorname{Isom}(M, g) \to \operatorname{Aut}(\mathfrak{g})$$

is a Lie group isomorphism. In particular, the identity component of $\operatorname{Isom}(M, g) \cong \operatorname{Aut}(\mathfrak{g})$ is the adjoint group G, which we know explicitly. It remains to find the other connected components of $\operatorname{Isom}(M, g) \cong \operatorname{Aut}(\mathfrak{g})$. [Gun10] shows that there are always finitely many connected components, and lists out the number of connected components for each Lie algebra $\mathfrak{g}$. However, [Gun10] does not explicitly describe these extra isometries. In this thesis, we explicitly describe these extra isometries.

As described above, each symmetric space can always be realised as a quotient space G/K. Nevertheless, a significant part of this thesis is devoted to describing a nice "geometric model" of the symmetric space corresponding to each Lie algebra $\mathfrak{g}$. For example, the symmetric space corresponding to the Lie algebra $\mathfrak{sl}(n, \mathbb{R})$ is the space of inner products on $\mathbb{R}^n$ with a fixed volume, and the symmetric space corresponding to the Lie algebra $\mathfrak{so}(p, q)$ is the space of q-dimensional subspaces of $\mathbb{R}^{p+q}$ which are negative-definite with respect to the standard (p, q)-form. Most of these models are readily found in the literature: see [Bes08, §10.K], [Hel78, §X Exercises D, §VIII Exercises C] or [BH99, §10]. The author believes that the geometric model corresponding to $\mathfrak{so}(n, \mathbb{C})$ is new, and the author thanks Juan Manuel Lorenzo Naveiro for his help.

1.3 Main result

The aim of this thesis is to prove the following theorem:

Theorem. *Let $\mathfrak{g}$ denote one of the following simple real Lie algebras of non-compact type: $\mathfrak{sl}(n, \mathbb{K})$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, $\mathfrak{so}(p, q)$, $\mathfrak{su}(p, q)$, $\mathfrak{sp}(p, q)$, $\mathfrak{sp}(2n, \mathbb{K})$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}$, and $\mathfrak{so}(n, \mathbb{K})$ for $\mathbb{K} = \mathbb{C}, \mathbb{H}$, with $\mathfrak{g} \neq \mathfrak{so}(4, 4), \mathfrak{so}(8, \mathbb{C})$. Tables 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, and 1.10 give*

(i) *the automorphism group $\operatorname{Aut}(\mathfrak{g})$ of $\mathfrak{g}$;*

(ii) *a model for the symmetric space of non-compact type (M, g) corresponding to $\mathfrak{g}$;*

(iii) *the full isometry group $\operatorname{Isom}(M, g)$ of (M, g).*

Notation

- $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ denote the real numbers, complex numbers, and quaternions, respectively. We let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ denote the basis for the imaginary quaternions. If $\lambda \in \mathbb{C}, \mathbb{H}$, then $\overline{\lambda}$ denotes the conjugate of λ.
- $\mathrm{M}_{p \times q}(\mathbb{K})$ denotes the set of all p by q matrices with entries in $\mathbb{K}$. We write $\mathrm{M}_n(\mathbb{K}) := \mathrm{M}_{n \times n}(\mathbb{K})$.
- If $X = (X_{ij}) \in \mathrm{M}_{p \times q}(\mathbb{K})$, then $X^\top$ is the matrix whose (i, j)th entry is X_{ji}, $\overline{X}$ is the matrix whose (i, j)th entry is $\overline{X_{ij}}$, and $X^* = \overline{X}^\top$.
- $\operatorname{Sym}(n, \mathbb{K}) := \{X \in \mathrm{M}_n(\mathbb{K}) : X^\top = X\}$, $\operatorname{Skew}(n, \mathbb{K}) := \{X \in \mathrm{M}_n(\mathbb{K}) : X^\top = -X\}$.
- $\operatorname{Herm}(n, \mathbb{K}) := \{X \in \mathrm{M}_n(\mathbb{K}) : X^* = X\}$. If $X \in \operatorname{Herm}(n, \mathbb{K})$, then $X > 0$ and $X < 0$ mean that X is positive-definite and negative-definite, respectively.
- $\operatorname{Re}(X)$ and $\operatorname{Im}(X)$ denotes the real and imaginary parts of X, respectively.
- I_n denotes the n by n identity matrix.

$\mathfrak{g}$	$\mathfrak{sl}(n,\mathbb{R})$ with $n \geq 2$
$\mathrm{Aut}(\mathfrak{g})$	Generated by $\mathrm{Ad}(\mathrm{SL}_{\pm}(n,\mathbb{R}))$ and $X \mapsto -X^{\top}$ (when $n \geq 3$), where $\mathrm{Ad} : \mathrm{SL}_{\pm}(n,\mathbb{R}) \to \mathrm{Aut}(\mathfrak{sl}(n,\mathbb{R}))$ is given by $\mathrm{Ad}(A)X = AXA^{-1}$. • Identity component is $\mathrm{Aut}(\mathfrak{g})^0 = \mathrm{Ad}(\mathrm{SL}(n,\mathbb{R}))$. • Kernel of Ad is $\pm\,\mathrm{I}_n$.
(M,g)	$M = \Big\{P \in \mathrm{Herm}(n,\mathbb{R}) : P > 0,\ \det(P) = 1\Big\},$ $g_P(V,W) = \mathrm{tr}(P^{-1}VP^{-1}W).$
$\mathrm{Isom}(M,g)$	Generated by $\tau(\mathrm{SL}_{\pm}(n,\mathbb{R}))$ and $P \mapsto P^{-1}$ (when $n \geq 3$), where $\tau : \mathrm{SL}_{\pm}(n,\mathbb{R}) \to \mathrm{Isom}(M,g)$ is given by $A \cdot P = APA^{\top}$. • Identity component is $\mathrm{Isom}(M,g)^0 = \tau(\mathrm{SL}(n,\mathbb{R}))$. • Kernel of τ is $\pm\,\mathrm{I}_n$. • Isotropy subgroup of τ at $\mathrm{I}_n \in M$ is $\mathrm{O}(n)$.

Table 1.1: Data for $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$.

$\mathfrak{g}$	$\mathfrak{sl}(n,\mathbb{C})$ with $n \geq 2$
$\mathrm{Aut}(\mathfrak{g})$	Generated by $\mathrm{Ad}(\mathrm{SL}(n,\mathbb{C}))$, $X \mapsto \overline{X}$, and $X \mapsto -X^{*}$ (when $n \geq 3$), where $\mathrm{Ad} : \mathrm{SL}(n,\mathbb{C}) \to \mathrm{Aut}(\mathfrak{sl}(n,\mathbb{C}))$ is given by $\mathrm{Ad}(A)X = AXA^{-1}$. • Identity component is $\mathrm{Aut}(\mathfrak{g})^0 = \mathrm{Ad}(\mathrm{SL}(n,\mathbb{C}))$. • Kernel of Ad is $\lambda\,\mathrm{I}_n$ such that $\lambda^n = 1$.
(M,g)	$M = \Big\{P \in \mathrm{Herm}(n,\mathbb{C}) : P > 0,\ \det(P) = 1\Big\},$ $g_P(V,W) = \mathrm{Re}\,\mathrm{tr}(P^{-1}VP^{-1}W).$
$\mathrm{Isom}(M,g)$	Generated by $\tau(\mathrm{SL}(n,\mathbb{C}))$, $P \mapsto \overline{P}$, and $P \mapsto P^{-1}$ (when $n \geq 3$), where $\tau : \mathrm{SL}(n,\mathbb{C}) \to \mathrm{Isom}(M,g)$ is given by $A \cdot P = APA^{*}$. • Identity component is $\mathrm{Isom}(M,g)^0 = \tau(\mathrm{SL}(n,\mathbb{C}))$. • Kernel of τ is $\lambda\,\mathrm{I}_n$ such that $\lambda^n = 1$. • Isotropy subgroup of τ at $\mathrm{I}_n \in M$ is $\mathrm{SU}(n)$.

Table 1.2: Data for $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$.

$\mathfrak{g}$	$\mathfrak{sl}(n,\mathbb{H})$ with $n \geq 2$
$\mathrm{Aut}(\mathfrak{g})$	Generated by $\mathrm{Ad}(\mathrm{SL}(n,\mathbb{H}))$ and $X \mapsto -X^*$, where $\mathrm{Ad} : \mathrm{SL}(n,\mathbb{H}) \to \mathrm{Aut}(\mathfrak{sl}(n,\mathbb{H}))$ is given by $\mathrm{Ad}(A)X = AXA^{-1}$. • Identity component is $\mathrm{Aut}(\mathfrak{g})^0 = \mathrm{Ad}(\mathrm{SL}(n,\mathbb{H}))$. • Kernel of Ad is $\pm\,\mathrm{I}_n$.
(M,g)	$M = \left\{P \in \mathrm{Herm}(n,\mathbb{H}) : P > 0,\ \det(\Psi(P)) = 1\right\}$, where $\Psi : \mathrm{M}_n(\mathbb{H}) \to \mathrm{M}_{2n}(\mathbb{C})$ is given by $A + \mathbf{j}C \mapsto \begin{pmatrix} A & -\overline{C} \\ C & \overline{A} \end{pmatrix}$, $g_P(V,W) = \mathrm{Re}\,\mathrm{tr}(P^{-1}VP^{-1}W)$.
$\mathrm{Isom}(M,g)$	Generated by $\tau(\mathrm{SL}(n,\mathbb{H}))$ and $P \mapsto P^{-1}$, where $\tau : \mathrm{SL}(n,\mathbb{H}) \to \mathrm{Isom}(M,g)$ is given by $A \cdot P = APA^*$. • Identity component is $\mathrm{Isom}(M,g)^0 = \tau(\mathrm{SL}(n,\mathbb{H}))$. • Kernel of τ is $\pm\,\mathrm{I}_n$. • Isotropy subgroup of τ at $\mathrm{I}_n \in M$ is $\mathrm{Sp}(n)$.

Table 1.3: Data for $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{H})$.

$\mathfrak{g}$	$\mathfrak{so}(p,q)$ with $p \geq q \geq 1$, $(p,q) \neq (1,1),(2,2),(4,4)$
$\mathrm{Aut}(\mathfrak{g})$	Generated by $\mathrm{Ad}(\mathrm{O}(p,q))$ and $X \mapsto \begin{pmatrix} 0 & \mathrm{I}_p \\ \mathrm{I}_p & 0 \end{pmatrix} X \begin{pmatrix} 0 & \mathrm{I}_p \\ \mathrm{I}_p & 0 \end{pmatrix}$ (when $p = q$), where $\mathrm{Ad} : \mathrm{O}(p,q) \to \mathrm{Aut}(\mathfrak{so}(p,q))$ is given by $\mathrm{Ad}(A)X = AXA^{-1}$. • Identity component is $\mathrm{Aut}(\mathfrak{g})^0 = \mathrm{Ad}(\mathrm{O}(p,q)^0)$. • Kernel of Ad is $\pm\,\mathrm{I}_{p+q}$.
(M,g)	$M = \left\{X \in \mathrm{M}_{p\times q}(\mathbb{R}) : X^\top X - \mathrm{I}_q < 0\right\}$, $g_X(V,W) = \mathrm{tr}((\mathrm{I}_p - XX^\top)^{-1}V(\mathrm{I}_q - X^\top X)^{-1}W^\top)$.
$\mathrm{Isom}(M,g)$	Generated by $\tau(\mathrm{O}(p,q))$ and $X \mapsto X^\top$ (when $p = q$), where $\tau : \mathrm{O}(p,q) \to \mathrm{Isom}(M,g)$ is given by $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot X = (AX+B)(CX+D)^{-1}$. • Identity component is $\mathrm{Isom}(M,g)^0 = \tau(\mathrm{O}(p,q)^0)$. • Kernel of τ is $\pm\,\mathrm{I}_n$. • Isotropy subgroup of τ at $0 \in M$ is $\mathrm{O}(p) \times \mathrm{O}(q)$.

Table 1.4: Data for $\mathfrak{g} = \mathfrak{so}(p,q)$.

$\mathfrak{g}$	$\mathfrak{su}(p,q)$ with $p \geq q \geq 1$
$\operatorname{Aut}(\mathfrak{g})$	Generated by $\operatorname{Ad}(\mathrm{SU}(p,q))$, $X \mapsto \overline{X}$, and $X \mapsto \left(\begin{smallmatrix} 0 & \mathrm{I}_p \\ \mathrm{I}_p & 0 \end{smallmatrix}\right) X \left(\begin{smallmatrix} 0 & \mathrm{I}_p \\ \mathrm{I}_p & 0 \end{smallmatrix}\right)$ (when $p = q \geq 2$), where $\operatorname{Ad} : \mathrm{SU}(p,q) \to \operatorname{Aut}(\mathfrak{su}(p,q))$ is given by $\operatorname{Ad}(A)X = AXA^{-1}$. • Identity component is $\operatorname{Aut}(\mathfrak{g})^0 = \operatorname{Ad}(\mathrm{SU}(p,q))$. • Kernel of Ad is $\lambda\,\mathrm{I}_{p+q}$ where $\lambda^{p+q} = 1$.
(M, g)	$M = \Big\{ Z \in \mathrm{M}_{p\times q}(\mathbb{C}) : Z^*Z - \mathrm{I}_q < 0 \Big\},$ $g_Z(V,W) = \operatorname{Re}\operatorname{tr}((\mathrm{I}_p - ZZ^*)^{-1} V (\mathrm{I}_q - Z^*Z)^{-1} W^*).$
$\operatorname{Isom}(M, g)$	Generated by $\tau(\mathrm{SU}(p,q))$, $Z \mapsto \overline{Z}$, and $Z \mapsto Z^*$ (when $p = q \geq 2$), where $\tau : \mathrm{SU}(p,q) \to \operatorname{Isom}(M,g)$ is given by $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \cdot Z = (AZ+B)(CZ+D)^{-1}$. • Identity component is $\operatorname{Isom}(M,g)^0 = \tau(\mathrm{SU}(p,q))$. • Kernel of τ is $\lambda\,\mathrm{I}_{p+q}$ where $\lambda^{p+q} = 1$. • Isotropy subgroup of τ at $0 \in M$ is $S(\mathrm{U}(p) \times \mathrm{U}(q))$.

Table 1.5: Data for $\mathfrak{g} = \mathfrak{su}(p,q)$.

$\mathfrak{g}$	$\mathfrak{sp}(p,q)$ with $p \geq q \geq 1$
$\operatorname{Aut}(\mathfrak{g})$	Generated by $\operatorname{Ad}(\mathrm{Sp}(p,q))$ and $X \mapsto \left(\begin{smallmatrix} 0 & \mathrm{I}_p \\ \mathrm{I}_p & 0 \end{smallmatrix}\right) X \left(\begin{smallmatrix} 0 & \mathrm{I}_p \\ \mathrm{I}_p & 0 \end{smallmatrix}\right)$ (when $p = q$), where $\operatorname{Ad} : \mathrm{Sp}(p,q) \to \operatorname{Aut}(\mathfrak{sp}(p,q))$ is given by $\operatorname{Ad}(A)X = AXA^{-1}$. • Identity component is $\operatorname{Aut}(\mathfrak{g})^0 = \operatorname{Ad}(\mathrm{Sp}(p,q))$. • Kernel of Ad is $\pm\,\mathrm{I}_{p+q}$.
(M, g)	$M = \Big\{ Z \in \mathrm{M}_{p\times q}(\mathbb{H}) : Z^*Z - \mathrm{I}_q < 0 \Big\},$ $g_Z(V,W) = \operatorname{Re}\operatorname{tr}((\mathrm{I}_p - ZZ^*)^{-1} V (\mathrm{I}_q - Z^*Z)^{-1} W^*).$
$\operatorname{Isom}(M, g)$	Generated by $\tau(\mathrm{Sp}(p,q))$ and $Z \mapsto Z^*$ (when $p = q$), where $\tau : \mathrm{Sp}(p,q) \to \operatorname{Isom}(M,g)$ is given by $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \cdot Z = (AZ+B)(CZ+D)^{-1}$. • Identity component is $\operatorname{Isom}(M,g)^0 = \tau(\mathrm{Sp}(p,q))$. • Kernel of τ is $\pm\,\mathrm{I}_{p+q}$. • Isotropy subgroup of τ at $0 \in M$ is $\mathrm{Sp}(p) \times \mathrm{Sp}(q)$.

Table 1.6: Data for $\mathfrak{g} = \mathfrak{su}(p,q)$.

$\mathfrak{g}$	$\mathfrak{sp}(2n,\mathbb{R})$ with $n \geq 1$
$\mathrm{Aut}(\mathfrak{g})$	Generated by $\mathrm{Ad}(\mathrm{Sp}(2n,\mathbb{R}))$ and $X \mapsto \left(\begin{smallmatrix} \mathrm{I}_n & 0 \\ 0 & -\mathrm{I}_n \end{smallmatrix}\right) X \left(\begin{smallmatrix} \mathrm{I}_n & 0 \\ 0 & -\mathrm{I}_n \end{smallmatrix}\right)$, where $\mathrm{Ad} : \mathrm{Sp}(2n,\mathbb{R}) \to \mathrm{Aut}(\mathfrak{sp}(2n,\mathbb{R}))$ is given by $\mathrm{Ad}(A)X = AXA^{-1}$. • Identity component is $\mathrm{Aut}(\mathfrak{g})^0 = \mathrm{Ad}(\mathrm{Sp}(2n,\mathbb{R}))$. • Kernel of Ad is $\pm\,\mathrm{I}_{2n}$.
(M,g)	$$M = \Big\{ Z \in \mathrm{Sym}(n,\mathbb{C}) : \ \mathrm{Im}(Z) > 0 \Big\},$$ $$g_{X+\mathbf{i}Y}(V,W) = \mathrm{Re}\,\mathrm{tr}(Y^{-1}VY^{-1}\overline{W}).$$
$\mathrm{Isom}(M,g)$	Generated by $\tau(\mathrm{Sp}(2n,\mathbb{R}))$ and $X + \mathbf{i}Y \mapsto -X + \mathbf{i}Y$, where $\tau : \mathrm{Sp}(2n,\mathbb{R}) \to \mathrm{Isom}(M,g)$ is given by $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \cdot Z = (AZ+B)(CZ+D)^{-1}$. • Identity component is $\mathrm{Isom}(M,g)^0 = \tau(\mathrm{Sp}(2n,\mathbb{R}))$. • Kernel of τ is $\pm\,\mathrm{I}_{2n}$. • Isotropy subgroup of τ at $\mathbf{i}\,\mathrm{I}_n \in M$ is $\Phi(\mathrm{U}(n))$, where $\Phi : \mathrm{M}_n(\mathbb{C}) \to \mathrm{M}_{2n}(\mathbb{R})$ is given by $A + \mathbf{i}C \mapsto \left(\begin{smallmatrix} A & -C \\ C & A \end{smallmatrix}\right)$.

Table 1.7: Data for $\mathfrak{g} = \mathfrak{sp}(2n,\mathbb{R})$.

$\mathfrak{g}$	$\mathfrak{sp}(2n,\mathbb{C})$ with $n \geq 1$
$\mathrm{Aut}(\mathfrak{g})$	Generated by $\mathrm{Ad}(\mathrm{Sp}(2n,\mathbb{C}))$ and $X \mapsto \overline{X}$, where $\mathrm{Ad} : \mathrm{Sp}(2n,\mathbb{C}) \to \mathrm{Aut}(\mathfrak{sp}(2n,\mathbb{C}))$ is given by $\mathrm{Ad}(A)X = AXA^{-1}$. • Identity component is $\mathrm{Aut}(\mathfrak{g})^0 = \mathrm{Ad}(\mathrm{Sp}(2n,\mathbb{C}))$. • Kernel of Ad is $\pm\,\mathrm{I}_{2n}$.
(M,g)	$$M = \Big\{ P \in \mathrm{Herm}(2n,\mathbb{C}) : P > 0, \ P \in \mathrm{Sp}(2n,\mathbb{C}) \Big\},$$ $$g_P(V,W) = \mathrm{Re}\,\mathrm{tr}(P^{-1}VP^{-1}W).$$
$\mathrm{Isom}(M,g)$	Generated by $\tau(\mathrm{Sp}(2n,\mathbb{C}))$ and $P \mapsto \overline{P}$, where $\tau : \mathrm{Sp}(2n,\mathbb{C}) \to \mathrm{Isom}(M,g)$ is given by $A \cdot P = APA^*$. • Identity component is $\mathrm{Isom}(M,g)^0 = \tau(\mathrm{Sp}(2n,\mathbb{C}))$. • Kernel of τ is $\pm\,\mathrm{I}_{2n}$. • Isotropy subgroup of τ at $\mathrm{I}_n \in M$ is $\Psi(\mathrm{Sp}(n))$, where $\Phi : \mathrm{M}_n(\mathbb{H}) \to \mathrm{M}_{2n}(\mathbb{C})$ is given by $A + \mathbf{j}C \mapsto \left(\begin{smallmatrix} A & -\overline{C} \\ C & \overline{A} \end{smallmatrix}\right)$.

Table 1.8: Data for $\mathfrak{g} = \mathfrak{sp}(2n,\mathbb{C})$.

$\mathfrak{g}$	$\mathfrak{so}(n,\mathbb{C})$ with $n \geq 3$, $n \neq 4, 8$
$\operatorname{Aut}(\mathfrak{g})$	Generated by $\operatorname{Ad}(\mathfrak{so}(n,\mathbb{C}))$ and $X \mapsto \overline{X}$, where $\operatorname{Ad} : \mathrm{O}(n,\mathbb{C}) \to \operatorname{Aut}(\mathrm{O}(n,\mathbb{C}))$ is given by $\operatorname{Ad}(A)X = AXA^{-1}$. • Identity component is $\operatorname{Aut}(\mathfrak{g})^0 = \operatorname{Ad}(\mathrm{SO}(n,\mathbb{C}))$. • Kernel of Ad is $\pm \mathrm{I}_n$.
(M, g)	$M = \left\{X \in \operatorname{Skew}(n,\mathbb{R}) : X^\top X - \mathrm{I}_n < 0\right\},$ $g_X(V, W) = \operatorname{tr}((\mathrm{I}_n - XX^\top)^{-1} V (\mathrm{I}_n - X^\top X)^{-1} W^\top).$
$\operatorname{Isom}(M, g)$	Generated by $\tau(\mathrm{O}(n,\mathbb{C}))$ and $X \mapsto X^\top$, where $\tau : \mathrm{O}(n,\mathbb{C}) \to \operatorname{Isom}(M, g)$ is given by $(A + \mathbf{i}C) \cdot X = (AX - C)(CX + A)^{-1}$. • Identity component is $\operatorname{Isom}(M, g)^0 = \tau(\mathrm{O}(n,\mathbb{C}))$. • Kernel of τ is $\pm \mathrm{I}_n$. • Isotropy subgroup of τ at $0 \in M$ is $\mathrm{O}(n)$.

Table 1.9: Data for $\mathfrak{g} = \mathfrak{so}(n,\mathbb{C})$.

$\mathfrak{g}$	$\mathfrak{so}(n,\mathbb{H})$ with $n \geq 5$
$\operatorname{Aut}(\mathfrak{g})$	Generated by $\operatorname{Ad}(\mathrm{SO}(n,\mathbb{H}))$ and $X \mapsto -\mathbf{j}X\mathbf{j}$, where $\operatorname{Ad} : \mathrm{SO}(n,\mathbb{H}) \to \operatorname{Aut}(\mathfrak{so}(n,\mathbb{H}))$ is given by $\operatorname{Ad}(A)X = AXA^{-1}$. • Identity component is $\operatorname{Aut}(\mathfrak{g})^0 = \operatorname{Ad}(\mathrm{SO}(n,\mathbb{H}))$. • Kernel of Ad is $\pm \mathrm{I}_n$.
(M, g)	$M = \left\{Z \in \operatorname{Skew}(n,\mathbb{C}) : Z^* Z - \mathrm{I}_n < 0\right\},$ $g_Z(V, W) = \operatorname{Re}\operatorname{tr}((\mathrm{I}_n - ZZ^*)^{-1} V (\mathrm{I}_n - Z^* Z)^{-1} W^*).$
$\operatorname{Isom}(M, g)$	Generated by $\tau(\mathrm{SO}(n,\mathbb{H}))$ and $Z \mapsto \overline{Z}$, where $\tau : \mathrm{O}(n,\mathbb{H}) \to \operatorname{Isom}(M, g)$ is given by $(A + \mathbf{j}C) \cdot X = (AX - \overline{C})(CX + \overline{A})^{-1}$. • Identity component is $\operatorname{Isom}(M, g)^0 = \tau(\mathrm{SO}(n,\mathbb{H}))$. • Kernel of τ is $\pm \mathrm{I}_n$. • Isotropy subgroup of τ at $0 \in M$ is $\mathrm{U}(n)$.

Table 1.10: Data for $\mathfrak{g} = \mathfrak{so}(n,\mathbb{H})$.

1.4 Further directions

1.4.1 The triality automorphism

We did not complete the cases when the Lie algebra is $\mathfrak{so}(4,4)$ or $\mathfrak{so}(8,\mathbb{C})$ due to time constraints. In both cases, we are missing just one automorphism of order three, called the *triality automorphism* [Gun10, Remark 2.7]. The triality automorphism of $\mathfrak{so}(4,4)$ is obtained by restricting the triality automorphism of $\mathfrak{so}(8,\mathbb{C})$, so we focus the latter.

For a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, the symmetries of the Dynkin diagram of $\mathfrak{g}$ give rise to complex automorphisms of $\mathfrak{g}$ [Kna96, §II.10 Example 2]. In fact, the group of complex automorphisms $\mathrm{Aut}_{\mathbb{C}}(\mathfrak{g})$ is a semidirect product of the adjoint group and symmetries of the Dynkin diagram [FH91, Proposition D.40].

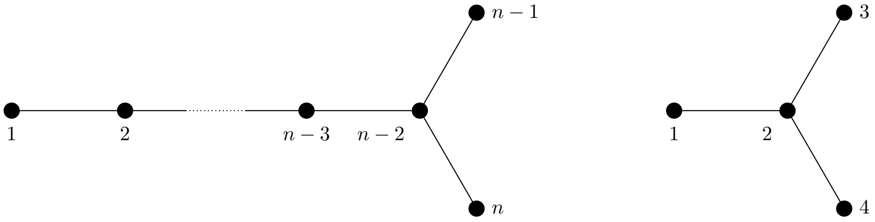

Figure 1.1: Dynkin diagrams for $\mathfrak{so}(2n,\mathbb{C})$ (left) and $\mathfrak{so}(8,\mathbb{C})$ (right).

When $n \geq 5$, the Dynkin diagram of $\mathfrak{so}(2n,\mathbb{C})$ only has one symmetry[3]: interchanging the nodes $n-1$ and n. When $n = 4$, there is an additional symmetry for the Dynkin diagram of $\mathfrak{so}(8,\mathbb{C})$, obtained from rotating the diagram (i.e. cycling the nodes 1, 3 and 4). This extra symmetry corresponds to the triality automorphism of $\mathfrak{so}(8,\mathbb{C})$.

1.4.2 Compact type

The author is also interested in the isometries of irreducible symmetric spaces of compact type. Duality gives a bijection between the simply-connected compact types and the non-compact types. Similar to the non-compact case, there are many well-known geometric models for the irreducible symmetric spaces of compact type [Bes08, §10.K]. The author is interested in understanding these models.

Let (M,g) denote an irreducible symmetric space of compact type. The identity component of $\mathrm{Isom}(M,g)$ is well understood. Moreover, the group of connected components of $\mathrm{Isom}(M,g)$ is also known: if (M',g') denotes the non-compact dual of (M,g), then the groups of connected components of the isometry groups are isomorphic [Loo69, §VII.4]:

$$\mathrm{Isom}(M,g)/\,\mathrm{Isom}(M,g)^0 \cong \mathrm{Isom}(M',g')/\,\mathrm{Isom}(M',g')^0.$$

---

[3] This symmetry corresponds to the automorphism of $\mathfrak{so}(2n,\mathbb{C})$ obtained from conjugation by some $A \in \mathrm{O}(2n,\mathbb{C})$ with $\det(A) = -1$ [FH91, Exercise 22.25].

1.5 Organisation of thesis

Chapter 2 is devoted to the Lie theoretic side of the thesis. The primary aim of §§2.1, 2.2, and 2.3 is to fix notation. A significant minority of the Lie algebras, Lie groups, and symmetric spaces of interest are best described with matrices and linear algebra over the quaternions, so we dedicate §2.1 to their study. In §2.2, we construct a number of matrix Lie groups and Lie algebras: most of the automorphisms and isometries are best described by the action of matrix Lie groups, as can be seen in the tables of §1.3. Moreover, the ten infinite families of simple Lie algebras of non-compact type are realised as matrix Lie algebras in this thesis. In §2.3, we review some basic topics about simple Lie algebras over $\mathbb{R}$, the most important of which are their classification (§2.3.4) and Cartan decomposition (§2.3.5). Finally, in §2.4, we compute the automorphism groups of the ten infinite families of simple Lie algebras of non-compact type.

Chapter 3 is devoted to the geometric side of the thesis. §3.1 recalls basic facts about irreducibility of Riemannian manifolds, isometry groups, and symmetric spaces. §3.2 describes the correspondence between symmetric spaces and simple Lie algebras over $\mathbb{R}$, particularly in the non-compact case. Finally, in §§3.3, 3.4, 3.5, 3.6, 3.7, and 3.8, we construct the geometric models of the ten infinite families of irreducible symmetric spaces of non-compact type, and compute their isometry groups.

Chapter 2

Lie groups and Lie algebras

Throughout this chapter, the reader is assumed to be familiar with the basics of Lie groups and Lie algebras. Some approachable references for these topics are [Hal15], [Lee13], [Tap05], or [EW06].

2.1 Quaternionic matrices

Three of the ten infinite families of simple Lie algebras of non-compact type are best described via quaternionic matrices. Moreover, the models (constructed in Chapter 3) for the symmetric spaces corresponding to $\mathfrak{sl}(n, \mathbb{K})$ and $\mathfrak{sp}(p, q)$ are described in terms of quaternionic linear algebra. In this section, we review some basic definitions and facts about quaternions, quaternionic matrices, quaternionic vector spaces, and embeddings of matrices. The main references for this section are [Tap05, §§1.2-1.5], [FP03], [Rod14], and [Kna96, §1.8].

2.1.1 Quaternions

The *quaternions* $\mathbb{H}$ is the four-dimensional (associative unital) algebra over $\mathbb{R}$ defined in the following manner. The underlying real vector space of $\mathbb{H}$ is the real vector space with basis $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, where $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ here are formal symbols. The multiplication of $\mathbb{H}$ is the unique $\mathbb{R}$-bilinear operation induced by the following table:

	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	$-\mathbf{1}$	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	$-\mathbf{1}$	$\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{i}$	$-\mathbf{1}$

For example, $\mathbf{ij} = \mathbf{k}$ and $\mathbf{ji} = -\mathbf{k}$. Thus, the multiplication on the quaternions is not commutative. A *quaternion* is an element of the quaternions $\mathbb{H}$. We identify the real numbers $\mathbb{R}$ and complex numbers $\mathbb{C}$ inside $\mathbb{H}$ via $a + b\mathbf{i} \leftrightarrow a\mathbf{1} + b\mathbf{i}$. Under this identification, every quaternion λ can be written uniquely as $\lambda = \alpha + \mathbf{j}\beta$, where $\alpha, \beta \in \mathbb{C}$.

Now, let $\lambda = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H}$. The *conjugate* of λ is the quaternion

$$\overline{\lambda} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}.$$

Thus, $\lambda \in \mathbb{R}$ if and only if $\lambda = \overline{\lambda}$. If λ and μ are quaternions, then $\overline{\mu\lambda} = \overline{\lambda}\,\overline{\mu}$. The *norm* of λ is the real number $|\lambda| = \sqrt{a^2 + b^2 + c^2 + d^2}$. The quaternions $\mathbb{H}$ form a *division ring* (also called a *skew-field*): every non-zero $\lambda \in \mathbb{H}$ has a multiplicative inverse, which is given by $\lambda^{-1} = \overline{\lambda}/|\lambda|^2$.

2.1.2 Quaternionic matrices

A *quaternionic matrix* is a matrix with entries in $\mathbb{H}$. Addition, multiplication, transposes $(\cdot)^\top$ and trace tr for matrices over $\mathbb{H}$ are defined in the same manner as over $\mathbb{R}$ or $\mathbb{C}$. However, due to the non-commutativity of the quaternions, we have $(XY)^\top \neq Y^\top X^\top$ and $\operatorname{tr}(XY) \neq \operatorname{tr}(YX)$ for general quaternionic matrices X and Y.

Given a quaternionic matrix X, the *conjugate* of X is the quaternionic matrix $\overline{X}$ given by quaternionic conjugation entry-wise: $(\overline{X})_{ij} := \overline{X_{ij}}$. In general, we have $\overline{XY} \neq \overline{X}\,\overline{Y}$ and $\overline{XY} \neq \overline{Y}\,\overline{X}$. The *conjugate transpose* of X is $X^* := \overline{X^\top} = (\overline{X})^\top$. The conjugate transpose satisfies

$$(XY)^* = Y^* X^*.$$

Let X be an n by n quaternionic matrix. We say that X is

(i) *Hermitian* if $X^* = X$. In this case, $v^* X v$ is a real number for any $v \in \mathbb{H}^n$. Here, we view v as an n by 1 matrix.

(ii) *skew-Hermitian* if $X^* = -X$.

(iii) *positive-definite* if X is Hermitian and $v^* X v > 0$ for all non-zero $v \in \mathbb{H}^n$. In this case, we write $X > 0$. *Negative-definite* matrices are defined analogously.

(iv) *invertible* if there exists an n by n quaternionic matrix Y such that $XY = YX = \mathrm{I}_n$. In this case, the matrix Y is unique, and we write $X^{-1} = Y$.

(v) *unitary* if X is invertible, and $X^{-1} = X^*$.

Remark 2.1 ([Tap05, Exercise 1.5]). The usual formula for determinants over $\mathbb{R}$ and $\mathbb{C}$ no longer detects invertibility if we generalise it to the quaternionic setting: for example, let

$$A = \begin{pmatrix} \mathbf{i} & \mathbf{j} \\ \mathbf{i} & \mathbf{j} \end{pmatrix}.$$

Then $\mathbf{ij} - \mathbf{ji} = 2\mathbf{k}$, but A is not invertible[1].

2.1.3 Quaternionic vector spaces

A *(right) vector space over* $\mathbb{H}$ is a right $\mathbb{H}$-module. The reason we want scalars acting from the right instead of the left is because we would like matrices $\mathrm{M}_n(\mathbb{H})$ to act $\mathbb{H}$-linearly on $\mathbb{H}^n$ from the left. More precisely, consider $\mathbb{H}^n$ as a right $\mathbb{H}$-module in the obvious way, and observe that the map $L_X : \mathbb{H}^n \to \mathbb{H}^n$ given by $v \mapsto Xv$ is $\mathbb{H}$-linear for each $X \in \mathrm{M}_n(\mathbb{H})$: $X(v\lambda) = (Xv)\lambda$ for $v \in \mathbb{H}^n$ and $\lambda \in \mathbb{H}$, thanks to associativity. However, if we instead consider $\mathbb{K}^n$ to be a left $\mathbb{H}$-module, then L_X would not be $\mathbb{H}$-linear in general because the quaternions are not commutative.

Nevertheless, the notions of linear maps, subspaces, spanning-sets, linear-independence, bases and dimension for vector spaces over $\mathbb{H}$ are defined and behave in the same manner as over $\mathbb{R}$ or $\mathbb{C}$. See [Art57, §1.2] for linear algebra over division rings. Other references for quaternionic vector spaces are [FP03, §4] or [Rod14, Chapter 3].

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[1] See [Asl96] for more about determinants and quaternionic matrices.

2.1.4 Embeddings

The references for this subsection are [Kna96, Pages 34-36] and [GW09, Exercise 1.10].

For $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, let $\mathrm{M}_{p\times q}(\mathbb{K})$ denote the collection of all p by q matrices with entries in $\mathbb{K}$. We consider $\mathrm{M}_{p\times q}(\mathbb{K})$ as a real vector space in the obvious way. We write square matrices $\mathrm{M}_n(\mathbb{K}) := \mathrm{M}_{n\times n}(\mathbb{K})$. Observe that

$$\mathrm{M}_{p\times q}(\mathbb{R}) \subseteq \mathrm{M}_{p\times q}(\mathbb{C}) \subseteq \mathrm{M}_{p\times q}(\mathbb{H}).$$

Conversely, we can embed complex matrices into real matrices of twice the size: let $\Phi : \mathrm{M}_n(\mathbb{C}) \hookrightarrow \mathrm{M}_{2n}(\mathbb{R})$ denote the injective $\mathbb{R}$-linear map given by

$$\Phi : A + \mathbf{i}C \mapsto \begin{pmatrix} A & -C \\ C & A \end{pmatrix}, \tag{2.1}$$

where $A, C \in \mathrm{M}_n(\mathbb{R})$. A matrix $Y \in \mathrm{M}_{2n}(\mathbb{R})$ belongs to the image of Φ if and only if

$$JY = YJ, \qquad J = \begin{pmatrix} 0 & \mathrm{I}_n \\ -\mathrm{I}_n & 0 \end{pmatrix}.$$

The map Φ enjoys the following properties whenever $X, Y \in \mathrm{M}_n(\mathbb{C})$:

$$\Phi(XY) = \Phi(X)\Phi(Y), \qquad \Phi(X^*) = \Phi(X)^\top, \qquad \operatorname{tr}\Phi(X) = 2\operatorname{Re}\operatorname{tr}(X). \tag{2.2}$$

It follows that $X \in \mathrm{M}_n(\mathbb{C})$ is Hermitian, skew-Hermitian, positive-definite, negative-definite, invertible, or unitary if and only if $\Phi(X) \in \mathrm{M}_{2n}(\mathbb{R})$ is so, respectively. Moreover, if we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ via $v + \mathbf{i}w \leftrightarrow (v, w)$ for $v, w \in \mathbb{R}^n$, then multiplication by X on $\mathbb{C}^n$ corresponds to multiplication by $\Phi(X)$ on $\mathbb{R}^{2n}$.

Similarly, we can embed quaternionic matrices into complex matrices of twice the size: let $\Psi : \mathrm{M}_n(\mathbb{H}) \hookrightarrow \mathrm{M}_{2n}(\mathbb{C})$ denote the injective $\mathbb{R}$-linear map given by

$$\Psi : A + \mathbf{j}C \mapsto \begin{pmatrix} A & -\overline{C} \\ C & \overline{A} \end{pmatrix}, \tag{2.3}$$

where $A, C \in \mathrm{M}_n(\mathbb{C})$. A matrix $Y \in \mathrm{M}_{2n}(\mathbb{C})$ belongs to the image of Ψ if and only if

$$JY = \overline{Y}J, \qquad J = \begin{pmatrix} 0 & \mathrm{I}_n \\ -\mathrm{I}_n & 0 \end{pmatrix}.$$

The map Ψ enjoys the following properties whenever $X, Y \in \mathrm{M}_n(\mathbb{H})$:

$$\Psi(XY) = \Psi(X)\Psi(Y), \qquad \Psi(X^*) = \Psi(X)^*, \qquad \operatorname{tr}\Psi(X) = 2\operatorname{Re}\operatorname{tr}(X). \tag{2.4}$$

It follows that $X \in \mathrm{M}_n(\mathbb{H})$ is Hermitian, skew-Hermitian, positive-definite, negative-definite, invertible, or unitary if and only if $\Psi(X) \in \mathrm{M}_{2n}(\mathbb{C})$ is so, respectively. Moreover, if we identify $\mathbb{H}^n$ with $\mathbb{C}^{2n}$ via $v + \mathbf{j}w \leftrightarrow (v, w)$ for $v, w \in \mathbb{C}^n$, then multiplication by X on $\mathbb{H}^n$ corresponds to multiplication by $\Psi(X)$ on $\mathbb{C}^{2n}$.

Remark 2.2. If $X \in \mathrm{M}_n(\mathbb{H})$, then $\det\Psi(X) \geq 0$. See [Zha97, Proposition 4.2].

2.2 Matrix Lie groups and Lie algebras

In this section, we construct the ten infinite families of simple Lie algebras (over $\mathbb{R}$) of non-compact type. Many of the automorphisms of these Lie algebras and the isometries of their corresponding symmetric spaces are best realised via an action of a matrix Lie group. We define these Lie groups in this section.

The main references for this section are [Ros02, Chapter 3], [GW09, §§1.1 and 1.2], [Kna96, §§1.8 and 1.14], and [Hel78, §X.2, Page 444].

2.2.1 General linear groups

For $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, the *general linear group* $\mathrm{GL}(n, \mathbb{K})$ is the set

$$\mathrm{GL}(n, \mathbb{K}) := \Big\{ A \in \mathrm{M}_n(\mathbb{K}) : A \text{ is invertible} \Big\},$$

equipped with matrix multiplication. When $\mathbb{K} = \mathbb{R}, \mathbb{C}$, we can write

$$\mathrm{GL}(n, \mathbb{K}) = \Big\{ A \in \mathrm{M}_n(\mathbb{K}) : \det(A) \neq 0 \Big\}.$$

When $\mathbb{K} = \mathbb{H}$ (see Remark 2.1), we have

$$\mathrm{GL}(n, \mathbb{H}) = \Big\{ A \in \mathrm{M}_n(\mathbb{H}) : \det \Psi(A) \neq 0 \Big\},$$

where $\Psi : \mathrm{M}_n(\mathbb{H}) \hookrightarrow \mathrm{M}_{2n}(\mathbb{C})$ is the embedding (2.3): recall that $A \in \mathrm{M}_n(\mathbb{H})$ is invertible if and only if $\Psi(A)$ is invertible. In any case, it follows that $\mathrm{GL}(n, \mathbb{K})$ is an open subset of $\mathrm{M}_n(\mathbb{K})$.

For $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, the *general linear algebra* $\mathfrak{gl}(n, \mathbb{K})$ is the real vector space $\mathrm{M}_n(\mathbb{K})$ equipped with the commutator bracket $[X, Y] = XY - YX$. The Lie algebra of $\mathrm{GL}(n, \mathbb{K})$ is $\mathfrak{gl}(n, \mathbb{K})$ in the following sense: The composition of the canonical isomorphisms

$$\mathrm{Lie}(\mathrm{GL}(n, \mathbb{K})) \longrightarrow T_{\mathrm{I}_n} \mathrm{GL}(n, \mathbb{K}) \longrightarrow \mathfrak{gl}(n, \mathbb{K})$$

is a Lie algebra isomorphism (see [Lee13, Propositions 8.41 and 8.48]). If G is a Lie subgroup of $\mathrm{GL}(n, \mathbb{K})$, then, via the isomorphism above, we identify the Lie algebra of G with

$$\mathfrak{g} = \Big\{ X \in \mathrm{GL}(n, \mathbb{K}) : \exp(tX) \in G \text{ for all } t \in \mathbb{R} \Big\}, \tag{2.5}$$

where $\exp : \mathfrak{gl}(n, \mathbb{K}) \to \mathrm{GL}(n, \mathbb{K})$ here denotes the matrix exponential [Lee13, Proposition 20.9].

2.2.2 Special linear groups

The *special linear group* $\mathrm{SL}(n, \mathbb{K})$ is defined as the kernel of $\det : \mathrm{GL}(n, \mathbb{K}) \to \mathbb{K}^\times$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $\det \circ \Psi : \mathrm{GL}(n, \mathbb{H}) \to \mathbb{R}^+$ for $\mathbb{K} = \mathbb{H}$:

$$\begin{aligned} \mathrm{SL}(n, \mathbb{K}) &:= \Big\{ A \in \mathrm{GL}(n, \mathbb{K}) : \det(A) = 1 \Big\} \qquad \mathbb{K} = \mathbb{R}, \mathbb{C}, \\ \mathrm{SL}(n, \mathbb{H}) &:= \Big\{ A \in \mathrm{GL}(n, \mathbb{K}) : \det \Psi(A) = 1 \Big\}. \end{aligned}$$

We also define

$$\mathrm{SL}_\pm(n, \mathbb{R}) := \Big\{ A \in \mathrm{GL}(n, \mathbb{R}) : \det(A) = \pm 1 \Big\}.$$

The special linear algebra $\mathfrak{sl}(n, \mathbb{K})$ is defined as

$$\begin{aligned} \mathfrak{sl}(n, \mathbb{K}) &:= \Big\{ X \in \mathfrak{gl}(n, \mathbb{K}) : \mathrm{tr}(X) = 0 \Big\} \qquad \mathbb{K} = \mathbb{R}, \mathbb{C}, \\ \mathfrak{sl}(n, \mathbb{H}) &:= \Big\{ X \in \mathfrak{gl}(n, \mathbb{K}) : \mathrm{Re}\,\mathrm{tr}(X) = 0 \Big\}. \end{aligned}$$

Proposition 2.3. *The Lie algebra of* $\mathrm{SL}(n, \mathbb{K})$ *is* $\mathfrak{sl}(n, \mathbb{K})$.

Proof. If $F : G \to H$ is a Lie group homomorphism, then

$$\mathrm{Lie}(\ker(F)) = \ker(F_*),$$

where $F_* : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ is the Lie algebra homomorphism induced by F. When $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the Lie algebra homomorphism induced by $\det : \mathrm{GL}(n, \mathbb{K}) \to \mathbb{K}^*$ is $\mathrm{tr} : \mathfrak{gl}(n, \mathbb{K}) \to \mathbb{K}$. When $\mathbb{K} = \mathbb{H}$, the Lie algebra homomorphism induced by $\det \circ \Psi : \mathrm{GL}(n, \mathbb{H}) \to \mathbb{R}^+$ is $2\,\mathrm{Re}\,\mathrm{tr} : \mathfrak{gl}(n, \mathbb{H}) \to \mathbb{R}$. ☐

2.2.3 Bilinear and sesquilinear forms

The remaining matrix Lie groups and Lie algebras of interest appear as automorphism groups of *non-degenerate bilinear* or *sesquilinear forms*. The aim of this subsection is to define these concepts.

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, and consider $\mathbb{K}^n$ as a (right) $\mathbb{K}$-vector space. Let $\varphi : \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}$ be a bi-additive map. We denote arbitrary elements of $\mathbb{K}^n$ by v and w, and arbitrary elements of $\mathbb{K}$ by λ and μ. We say that φ is *non-degenerate* if $\varphi(v, w) = 0$ for all w implies that $v = 0$.

We say that φ is a *bilinear form* if $\varphi(v\lambda, w\mu) = \varphi(v, w)\lambda\mu$. A bilinear form φ is called

(i) *symmetric* if $\varphi(w, v) = \varphi(v, w)$, and

(ii) *skew-symmetric* if $\varphi(w, v) = -\varphi(v, w)$.

If φ is a bilinear form, then there exists a unique matrix $F \in \mathrm{M}_n(\mathbb{K})$ such that

$$\varphi(v, w) = v^\top F w. \tag{2.6}$$

In this case, φ is symmetric, skew-symmetric, or non-degenerate if and only if F is symmetric, skew-symmetric, or invertible, respectively. It follows that every bilinear form can be written uniquely as a sum of a symmetric bilinear form and a skew-symmetric bilinear form.

Next, we say that the bi-additive map $\varphi : \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}$ is a *sesquilinear form* if $\varphi(v\lambda, w\mu) = \overline{\lambda}\varphi(v, w)\mu$. A sesquilinear form φ is called

(i) *Hermitian* if $\varphi(w, v) = \overline{\varphi(v, w)}$, and

(ii) *skew-Hermitian* if $\varphi(w, v) = -\overline{\varphi(v, w)}$.

If φ is a sesquilinear form, then there exists a unique matrix $F \in \mathrm{M}_n(\mathbb{K})$ such that

$$\varphi(v, w) = v^* F w. \tag{2.7}$$

In this case, φ is Hermitian, skew-Hermitian, or non-degenerate if and only if F is Hermitian, skew-Hermitian, or invertible respectively. It follows that every sesquilinear form can be written uniquely as a sum of a Hermitian sesquilinear form and skew-Hermitian sesquilinear form.

Two forms φ and φ' on $\mathbb{K}^n$ are called *equivalent* if there exists $A \in \mathrm{GL}(n, \mathbb{K})$ such that

$$\varphi'(\cdot, \cdot) = \varphi(A\cdot, A\cdot).$$

We now classify the non-degenerate symmetric, skew-symmetric, Hermitian, and skew-Hermitian forms on $\mathbb{K}^n$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, up to equivalence. First, observe the following:

(i) When $\mathbb{K} = \mathbb{R}$, the definitions of bilinear and sesquilinear forms agree.

(ii) When $\mathbb{K} = \mathbb{C}$, every skew-Hermitian form can be written as $i\varphi$ for some Hermitian form φ.

(iii) There are no non-zero bilinear forms over $\mathbb{H}$.

Proposition 2.4 ([Ros02, Appendix to §3.1, Pages 104-107])**.** *Let $\varphi : \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}$ be a non-degenerate bilinear or sesquilinear form.*

(i) *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, and let φ be Hermitian. Then there exist unique non-negative integers p and q with $p + q = n$ such that φ is equivalent to*

$$h_{p,q}(v, w) := v^* \mathrm{I}_{p,q}\, w, \qquad \mathrm{I}_{p,q} := \begin{pmatrix} \mathrm{I}_p & 0 \\ 0 & -\mathrm{I}_q \end{pmatrix}.$$

(ii) *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, and let φ be skew-symmetric. Then $n = 2m$ is even, and φ is equivalent to*

$$\omega(v, w) := v^\top J w, \qquad J := \begin{pmatrix} 0 & \mathrm{I}_m \\ -\mathrm{I}_m & 0 \end{pmatrix}.$$

(iii) *Let $\mathbb{K} = \mathbb{C}$, and let φ be symmetric. Then φ is equivalent to*

$$b(v, w) := v^\top w.$$

(iv) *Let $\mathbb{K} = \mathbb{H}$, and let φ be skew-Hermitian. Then φ is equivalent to*

$$c(v, w) := v^* \mathbf{i} w.$$

	sym.	skew-sym.	Herm.	skew-Herm.
$\mathbb{R}$	–	ω	$h_{p,q}$	–
$\mathbb{C}$	b	ω	$h_{p,q}$	–
$\mathbb{H}$	–	–	$h_{p,q}$	c

Table 2.1: Classification of non-degenerate forms

2.2.4 Automorphism groups of forms

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, and let $\varphi : \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}$ be a non-degenerate bilinear or sesquilinear form. Let F be the matrix associated with φ via (2.6) or (2.7). The *autormorphism group of φ* is the subgroup of $\mathrm{GL}(n, \mathbb{K})$ defined by

$$\mathrm{Aut}(\varphi) := \Big\{ A \in \mathrm{GL}(n, \mathbb{K}) : \varphi(A\cdot, A\cdot) = \varphi(\cdot, \cdot) \Big\}.$$

The *automorphism algebra of φ* is the subalgebra of $\mathfrak{gl}(n, \mathbb{K})$ defined by

$$\mathfrak{aut}(\varphi) := \Big\{ X \in \mathfrak{gl}(n, \mathbb{K}) : \varphi(X\cdot, \cdot) + \varphi(\cdot, X\cdot) = 0 \Big\}.$$

If φ is bilinear, then

$$\mathrm{Aut}(\varphi) = \Big\{ A \in \mathrm{GL}(n, \mathbb{K}) : A^\top F A = F \Big\},$$
$$\mathfrak{aut}(\varphi) = \Big\{ X \in \mathfrak{gl}(n, \mathbb{K}) : A^\top F + F A = 0 \Big\}.$$

If φ is sesquilinear, then

$$\mathrm{Aut}(\varphi) = \Big\{ A \in \mathrm{GL}(n, \mathbb{K}) : A^* F A = F \Big\},$$
$$\mathfrak{aut}(\varphi) = \Big\{ X \in \mathfrak{gl}(n, \mathbb{K}) : A^* F + F A = 0 \Big\}.$$

In either case, it follows that $\mathrm{Aut}(\varphi)$ is a closed subgroup of $\mathrm{GL}(n, \mathbb{K})$.

Proposition 2.5. *The Lie algebra of* $\mathrm{Aut}(\varphi)$ *is* $\mathfrak{aut}(\varphi)$.

Proof. Assume φ is sesquilinear. The proof when φ is bilinear is identical. First, suppose X belongs to $\mathrm{Lie}(\mathrm{Aut}(\varphi))$. By (2.5), it follows that $\exp(tX)^* F \exp(tX) = F$ for all $t \in \mathbb{R}$. Taking the differential at $t = 0$ gives $X^*F + FX = 0$, as desired.

Conversely, suppose X belongs to $\mathfrak{aut}(\varphi)$. Fix any $t \in \mathbb{R}$, and set $Y := tX$. By (2.5), we are done if we show that $\exp(Y)$ belongs to $\mathrm{Aut}(\varphi)$. Rearranging $Y^*F + FY = 0$ gives $0 = F^{-1}YF + Y$, since F is invertible. Therefore,

$$\mathrm{I}_n = \exp(F^{-1}Y^*F + Y) = F^{-1}\exp(Y)^*F\exp(Y),$$

where the last equality holds because $F^{-1}Y^*F$ and Y commute. $\square$

Proposition 2.6. *Let* φ *and* ψ *be two bilinear or sesquilinear forms associated with the matrices* F *and* F'*, respectively. Suppose* $\psi(\cdot,\cdot) = \varphi(A\cdot, A\cdot)$ *for some* $A \in \mathrm{GL}(n,\mathbb{K})$*. This condition is equivalent to* $F' = A^\top FA$ *if* φ *is bilinear, and* $F' = A^*FA$ *if* φ *is sesquilinear. Moreover,*

$$\mathrm{Aut}(\psi) = A^{-1}\,\mathrm{Aut}(\varphi)A, \qquad \mathfrak{aut}(\psi) = A^{-1}\mathfrak{aut}(\varphi)A.$$

Remark 2.7. Let $\mathbb{K} = \mathbb{R},\mathbb{C}$, and let $\lambda \in \mathbb{K}$ be non-zero. Then $\mathrm{Aut}(\lambda\varphi) = \mathrm{Aut}(\varphi)$, and $\mathfrak{aut}(\lambda\varphi) = \mathfrak{aut}(\varphi)$.

Let $\mathrm{I}_{p,q}$, J, $h_{p,q}$, ω, b and c be the matrices and forms defined in Proposition 2.4. We define

$$\begin{aligned}
\mathrm{O}(p,q) &:= \mathrm{Aut}(h^{\mathbb{R}}_{p,q}) = \Big\{A \in \mathrm{GL}(p+q,\mathbb{R}) : A^\top\,\mathrm{I}_{p,q}\,A = \mathrm{I}_{p,q}\Big\},\\
\mathrm{U}(p,q) &:= \mathrm{Aut}(h^{\mathbb{C}}_{p,q}) = \Big\{A \in \mathrm{GL}(p+q,\mathbb{C}) : A^*\,\mathrm{I}_{p,q}\,A = \mathrm{I}_{p,q}\Big\},\\
\mathrm{Sp}(p,q) &:= \mathrm{Aut}(h^{\mathbb{H}}_{p,q}) = \Big\{A \in \mathrm{GL}(p+q,\mathbb{H}) : A^*\,\mathrm{I}_{p,q}\,A = \mathrm{I}_{p,q}\Big\},\\
\mathrm{Sp}(2n,\mathbb{K}) &:= \mathrm{Aut}(\omega^{\mathbb{K}}) = \Big\{A \in \mathrm{GL}(2n,\mathbb{K}) : A^\top JA = J\Big\} \qquad \text{for } \mathbb{K} = \mathbb{R},\mathbb{C},\\
\mathrm{O}(n,\mathbb{C}) &:= \mathrm{Aut}(b) = \Big\{A \in \mathrm{GL}(n,\mathbb{C}) : A^\top A = \mathrm{I}_n\Big\},\\
\mathrm{SO}(n,\mathbb{H}) &:= \mathrm{Aut}(c) = \Big\{A \in \mathrm{GL}(n,\mathbb{H}) : A^*\mathbf{i}A = \mathbf{i}\,\mathrm{I}_n\Big\}.
\end{aligned}$$

	sym.	skew-sym.	Herm.	skew-Herm.
$\mathbb{R}$	–	$\mathrm{Sp}(2n,\mathbb{R})$	$\mathrm{O}(p,q)$	–
$\mathbb{C}$	$\mathrm{O}(n,\mathbb{C})$	$\mathrm{Sp}(2n,\mathbb{C})$	$\mathrm{U}(p,q)$	–
$\mathbb{H}$	–	–	$\mathrm{Sp}(p,q)$	$\mathrm{SO}(n,\mathbb{H})$

Table 2.2: Automorphism groups of non-degenerate forms

The following groups are also of interest:

$$\begin{aligned}
&\mathrm{O}(n) := \mathrm{O}(n,0), && \mathrm{U}(n) := \mathrm{U}(n,0), && \mathrm{Sp}(n) := \mathrm{Sp}(n,0),\\
&\mathrm{SO}(n) := \mathrm{O}(n) \cap \mathrm{SL}(n,\mathbb{R}), && \mathrm{SO}(n,\mathbb{C}) := \mathrm{O}(n,\mathbb{C}) \cap \mathrm{SL}(n,\mathbb{C}),\\
&\mathrm{SU}(p,q) := \mathrm{U}(p,q) \cap \mathrm{SL}(n,\mathbb{C}), && \mathrm{SU}(n) := \mathrm{U}(n) \cap \mathrm{SL}(n,\mathbb{C}).
\end{aligned}$$

We now consider the Lie algebras of these matrix Lie groups:

$$\mathfrak{so}(p,q) := \mathfrak{aut}(h^{\mathbb{R}}_{p,q}) = \Big\{A \in \mathfrak{gl}(p+q,\mathbb{R}) : A^\top\,\mathrm{I}_{p,q} + \mathrm{I}_{p,q}\,A = 0\Big\},$$

$$\begin{aligned}
\mathfrak{u}(p,q) &:= \mathfrak{aut}(h^{\mathbb{C}}_{p,q}) = \Big\{A \in \mathfrak{gl}(p+q,\mathbb{C}) : A^* \,\mathrm{I}_{p,q} + \mathrm{I}_{p,q}\, A = 0\Big\},\\
\mathfrak{sp}(p,q) &:= \mathfrak{aut}(h^{\mathbb{H}}_{p,q}) = \Big\{A \in \mathfrak{gl}(p+q,\mathbb{H}) : A^* \,\mathrm{I}_{p,q} + \mathrm{I}_{p,q}\, A = 0\Big\},\\
\mathfrak{sp}(2n,\mathbb{K}) &:= \mathfrak{aut}(\omega^{\mathbb{K}}) = \Big\{A \in \mathfrak{gl}(2n,\mathbb{K}) : A^\top J + JA = 0\Big\} \qquad \text{for } \mathbb{K} = \mathbb{R},\mathbb{C},\\
\mathfrak{so}(n,\mathbb{C}) &:= \mathfrak{aut}(b) = \Big\{A \in \mathfrak{gl}(n,\mathbb{C}) : A^\top + A = 0\Big\},\\
\mathfrak{so}(n,\mathbb{H}) &:= \mathfrak{aut}(c) = \Big\{A \in \mathfrak{gl}(n,\mathbb{H}) : A^* \mathbf{i} + \mathbf{i}A = 0\Big\}.
\end{aligned}$$

The following Lie algebras are also of interest:

$$\begin{aligned}
\mathfrak{so}(n) &:= \mathfrak{so}(n,0), & \mathfrak{u}(n) &:= \mathfrak{u}(n,0), & \mathfrak{sp}(n) &:= \mathfrak{sp}(n,0),\\
\mathfrak{su}(p,q) &:= \mathfrak{u}(p,q) \cap \mathfrak{sl}(n,\mathbb{C}), & \mathfrak{su}(n) &:= \mathfrak{u}(n) \cap \mathfrak{sl}(n,\mathbb{C}).
\end{aligned}$$

Note that if G and H are Lie subgroups of $\mathrm{GL}(n,\mathbb{K})$, then $\mathrm{Lie}(G\cap H) = \mathrm{Lie}(G) \cap \mathrm{Lie}(H)$.

2.2.5 Embeddings and isomorphisms

The aim of this subsection is to record some useful isomorphisms of Lie groups and Lie algebras.

The embeddings $\Phi : \mathrm{M}_n(\mathbb{C}) \hookrightarrow \mathrm{M}_{2n}(\mathbb{R})$ and $\Psi : \mathrm{M}_n(\mathbb{H}) \hookrightarrow \mathrm{M}_{2n}(\mathbb{C})$ defined by (2.1) and (2.3) restrict to group homomorphisms

$$\Phi : \mathrm{GL}(n,\mathbb{C}) \hookrightarrow \mathrm{GL}(2n,\mathbb{R}), \qquad \Psi : \mathrm{GL}(n,\mathbb{H}) \hookrightarrow \mathrm{GL}(2n,\mathbb{C}).$$

The induced Lie algebra homomorphisms are also given by Φ and Ψ, respectively:

$$\Phi : \mathfrak{gl}(n,\mathbb{C}) \hookrightarrow \mathfrak{gl}(2n,\mathbb{R}), \qquad \Psi : \mathfrak{gl}(n,\mathbb{H}) \hookrightarrow \mathfrak{gl}(2n,\mathbb{C}).$$

Proposition 2.8. *We have*

$$\begin{aligned}
\Phi(\mathrm{O}(n,\mathbb{C})) &= \mathrm{O}(n,n) \cap \mathrm{Aut}(k^{\mathbb{R}}),\\
\Phi(\mathrm{U}(n)) &= \mathrm{O}(2n) \cap \mathrm{Sp}(2n,\mathbb{R}),\\
\Psi(\mathrm{SO}(n,\mathbb{H})) &= \mathrm{U}(n,n) \cap \mathrm{Aut}(k^{\mathbb{C}}),\\
\Psi(\mathrm{Sp}(p,q)) &= \mathrm{Aut}(h') \cap \mathrm{Aut}(\omega'),
\end{aligned}$$

where $k^{\mathbb{K}} : \mathbb{K}^{2n} \times \mathbb{K}^{2n} \to \mathbb{K}$ *is the symmetric bilinear form given by*

$$k^{\mathbb{K}}(v,w) := v^\top \begin{pmatrix} 0 & \mathrm{I}_n \\ \mathrm{I}_n & 0 \end{pmatrix} w,$$

$h' : \mathbb{C}^{2(p+q)} \times \mathbb{C}^{2(p+q)} \to \mathbb{C}$ *is the Hermitian form given by*

$$h'(v,w) := v^* \begin{pmatrix} \mathrm{I}_{p,q} & 0 \\ 0 & \mathrm{I}_{p,q} \end{pmatrix} w,$$

and $\omega' : \mathbb{C}^{2(p+q)} \times \mathbb{C}^{2(p+q)} \to \mathbb{C}$ *is the skew-symmetric bilinear form given by*

$$\omega'(v,w) := v^\top \begin{pmatrix} 0 & \mathrm{I}_{p,q} \\ \mathrm{I}_{p,q} & 0 \end{pmatrix} w.$$

The analogous statements for their Lie algebras is also true.

Proof. Let us show $\Psi(\mathrm{SO}(n,\mathbb{H}) = \mathrm{U}(n,n) \cap \mathrm{Aut}(k^{\mathbb{C}})$. The proofs of the other equalities are analogous.

First, fix $Y \in \mathrm{U}(n,n) \cap \mathrm{Aut}(k^{\mathbb{C}})$, and let us show that Y belongs to the image of Ψ. Indeed, since $Y^* I_{p,q} Y = I_{p,q}$ and $Y^\top F Y = F$ where F is the matrix associated to $k = k^{\mathbb{C}}$, rearranging gives $JY = \overline{Y} J$.

It now suffices to show that $A \in \mathrm{GL}(n,\mathbb{H})$ preserves the skew-Hermitian form $c : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H}$ if and only if $\Psi(A)$ preserves $h_{n,n}$ and k. Let $\rho : \mathbb{H}^n \to \mathbb{C}^{2n}$ denote the identification $v + \mathbf{j}v' \mapsto (v, v')$. A straightforward computation shows that

$$c(v,w) = \mathbf{i} h_{n,n}(\rho(v), \rho(w)) + \mathbf{k} k(\rho(v), \rho(w)),$$

as desired. Also see [GW09, Exercises 1.11 and 1.12]. □

Remark 2.9. In the literature, it is common to see the following definitions:

$$\begin{aligned}
\mathrm{U}^*(2n) &:= \Psi(\mathrm{GL}(n,\mathbb{H})),\\
\mathrm{SU}^*(2n) &:= \Psi(\mathrm{SL}(n,\mathbb{H})) = \mathrm{U}^*(2n) \cap \mathrm{SL}(2n,\mathbb{C}),\\
\mathrm{SO}^*(2n) &:= \Psi(\mathrm{SO}(n,\mathbb{H})) = \mathrm{U}(n,n) \cap \mathrm{Aut}(k^{\mathbb{C}}),
\end{aligned}$$

with their Lie algebras $\mathfrak{u}^*(2n)$, $\mathfrak{su}^*(2n)$ and $\mathfrak{so}^*(2n)$ defined analogously. See [Hel78, §X.2, Page 444] or [Kna96, Pages 34, 36 and 70].

2.2.6 Block matrix forms

The following characterisations are useful:

$$\begin{aligned}
\mathrm{O}(p,q) &= \left\{ L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}(p+q,\mathbb{R}) : L^{-1} = \begin{pmatrix} A^\top & -C^\top \\ -B^\top & D^\top \end{pmatrix} \right\},\\
\mathrm{U}(p,q) &= \left\{ L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}(p+q,\mathbb{C}) : L^{-1} = \begin{pmatrix} A^* & -C^* \\ -B^* & D^* \end{pmatrix} \right\},\\
\mathrm{Sp}(p,q) &= \left\{ L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}(p+q,\mathbb{H}) : L^{-1} = \begin{pmatrix} A^* & -C^* \\ -B^* & D^* \end{pmatrix} \right\},\\
\mathrm{Sp}(2n,\mathbb{K}) &= \left\{ L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}(2n,\mathbb{K}) : L^{-1} = \begin{pmatrix} D^\top & -B^\top \\ -C^\top & A^\top \end{pmatrix} \right\}, \qquad \mathbb{K} = \mathbb{R}, \mathbb{C},
\end{aligned}$$

On the Lie algebra level, we can write

$$\begin{aligned}
\mathfrak{so}(p,q) &= \left\{ \begin{pmatrix} X & Y \\ Y^\top & W \end{pmatrix} : X \in \mathfrak{so}(p),\ W \in \mathfrak{so}(q),\ Y \in \mathrm{M}_{p\times q}(\mathbb{R}) \right\},\\
\mathfrak{u}(p,q) &= \left\{ \begin{pmatrix} X & Y \\ Y^* & W \end{pmatrix} : X \in \mathfrak{u}(p),\ W \in \mathfrak{u}(q),\ Y \in \mathrm{M}_{p\times q}(\mathbb{C}) \right\},\\
\mathfrak{sp}(p,q) &= \left\{ \begin{pmatrix} X & Y \\ Y^* & W \end{pmatrix} : X \in \mathfrak{sp}(p),\ W \in \mathfrak{sp}(q),\ Y \in \mathrm{M}_{p\times q}(\mathbb{H}) \right\},\\
\mathfrak{sp}(2n,\mathbb{K}) &= \left\{ \begin{pmatrix} X & Y \\ Z & -X^\top \end{pmatrix} : X \in \mathrm{M}_n(\mathbb{K}),\ Y, Z \in \mathrm{Sym}(n,\mathbb{K}) \right\}, \qquad \mathbb{K} = \mathbb{R}, \mathbb{C}\\
\mathfrak{so}(n,\mathbb{H}) &= \left\{ X + \mathbf{j}Y : X \in \mathfrak{u}(n),\ Y \in \mathfrak{so}(n,\mathbb{C}) \right\}.
\end{aligned}$$

2.2.7 Connectivity

In this subsection, we determine the connected components of the following Lie groups:

$$\mathrm{SL}(n,\mathbb{K}),\ \mathrm{SL}_\pm(n,\mathbb{R}),\ \mathrm{O}(p,q),\ \mathrm{SU}(p,q),\ \mathrm{Sp}(p,q),\ \mathrm{Sp}(2n,\mathbb{K}),\ \mathrm{O}(n,\mathbb{C}),\ \mathrm{SO}(n,\mathbb{H}). \tag{2.8}$$

Proposition 2.10 ([Kna96, Proposition 1.124])**.** *The following facts are true:*

(i) *The Lie groups* $\mathrm{SL}(n,\mathbb{K})$ *for* $\mathbb{K}=\mathbb{R},\mathbb{C},\mathbb{H}$, $\mathrm{SU}(p,q)$, $\mathrm{Sp}(p,q)$, $\mathrm{Sp}(2n,\mathbb{K})$ *for* $\mathbb{K}=\mathbb{R},\mathbb{C}$, *and* $\mathrm{SO}(n,\mathbb{H})$ *are connected.*

(ii) *The Lie group* $\mathrm{SL}_\pm(n,\mathbb{R})$ *has two connected components: its identity component is* $\mathrm{SL}(n,\mathbb{R})$*, and the other component consists of matrices in* $\mathrm{GL}(n,\mathbb{R})$ *with determinant* -1*.*

(iii) *The Lie group* $\mathrm{O}(n,\mathbb{C})$ *has two connected components: its identity component is* $\mathrm{SO}(n,\mathbb{C})$*, and the other component consists of matrices in* $\mathrm{O}(n,\mathbb{C})$ *with determinant* -1*.*

(iv) *Finally, the group* $\mathrm{O}(p,q)$ *has four connected components. If* $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ *is an element of* $\mathrm{O}(p,q)$*, then* A *and* D *are invertible, and the connected component of this element is determined by the signs of* $\det(A)$ *and* $\det(D)$*. The identity component is* $\mathrm{O}(p,q)^0$ *is characterised by* $\det(A)>0$ *and* $\det(D)>0$*.*

Proof. First, we can view each Lie group G' listed in (2.8) inside $\mathrm{GL}(m,\mathbb{C})$: if G' is defined via quaternionic matrices, embed G' inside $\mathrm{GL}(m,\mathbb{C})$ via Ψ. In any case, G' is subgroup of $\mathrm{GL}(m,\mathbb{C})$ equal the zero set of some real-valued polynomials in the real and imaginary parts of the matrix entries. Moreover, each G' is closed under conjugate transpose. Therefore, [Kna96, Proposition 1.122] tells us that the map

$$K'\times\mathfrak{p}\to G', \qquad (A,X)\mapsto A\exp(X)$$

is a homeomorphism, where $K':=G'\cap\mathrm{U}(m)$, and $\mathfrak{p}:=\mathrm{Lie}(G')\cap\mathrm{Herm}(m,\mathbb{C})$. Thus, it suffices to determine the connected components of K' for each G' listed in (2.8). Table 2.3 lists out all the possibilities for $K'=G'\cap\{\text{unitary matrices}\}$. For $n,p,q\geq 1$, we know the following:

(i) The Lie groups $\mathrm{SO}(n)$, $\mathrm{U}(n)$, $\mathrm{SU}(n)$, and $\mathrm{Sp}(n)$ are connected [Kna96, Proposition 1.115].

(ii) The Lie group

$$S(\mathrm{U}(p)\times\mathrm{U}(q)) := \Big\{(A,B) : A\in\mathrm{U}(p),\ B\in\mathrm{U}(q),\ \det(A)\det(B)=1\Big\}$$

is connected [Hel78, Chapter X.2, Page 449].

(iii) The Lie group $\mathrm{O}(n)$ has two connected components: its identity component is $\mathrm{SO}(n)$, and the other component consists of matrices in $\mathrm{O}(n)$ with determinant -1.

Proposition 2.10 now immediately follows from Table 2.3. ∎

G'	K'
$\mathrm{SL}_{\pm}(n,\mathbb{R})$	$\mathrm{O}(n)$
$\mathrm{SL}(n,\mathbb{C})$	$\mathrm{SU}(n)$
$\mathrm{SL}(n,\mathbb{H})$	$\mathrm{Sp}(n)$
$\mathrm{O}(p,q)$	$\left\{\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : A \in \mathrm{O}(p),\ D \in \mathrm{O}(q)\right\} \cong \mathrm{O}(p)\times\mathrm{O}(q)$
$\mathrm{SU}(p,q)$	$\left\{\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : \begin{matrix} A \in \mathrm{U}(p),\ D \in \mathrm{U}(q) \\ \det(A)\det(D) = 1 \end{matrix}\right\} \cong S(\mathrm{U}(p)\times\mathrm{U}(q))$
$\mathrm{Sp}(p,q)$	$\left\{\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : A \in \mathrm{Sp}(p),\ D \in \mathrm{Sp}(q)\right\} \cong \mathrm{Sp}(p)\times\mathrm{Sp}(q)$
$\mathrm{Sp}(2n,\mathbb{R})$	$\Phi(\mathrm{U}(n))$
$\mathrm{Sp}(2n,\mathbb{C})$	$\Psi(\mathrm{Sp}(n))$
$\mathrm{O}(n,\mathbb{C})$	$\mathrm{O}(n)$
$\mathrm{SO}(n,\mathbb{H})$	$\mathrm{U}(n)$

Table 2.3: Intersection of G' with unitary matrices. Here, Φ and Ψ denote the embeddings defined in §2.1.4.

2.3 Simple Lie algebras over $\mathbb{R}$

The aims of this section are to describe the classification of simple Lie algebras over $\mathbb{R}$ (§2.3.4) and to compute the Cartan decompositions of the ten infinite families of simple Lie algebras of non-compact type (§2.3.5). To do so, we first compute the complexifications (§2.3.1) and Killing forms (§2.3.2) of these Lie algebras.

2.3.1 Complexification

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$. The *complexification of* $\mathfrak{g}$, denoted by $\mathfrak{g}^{\mathbb{C}}$, is the Lie algebra over $\mathbb{C}$ obtained in the following manner. As an abelian group, we set $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \oplus \mathbf{i}\mathfrak{g}$, where $\mathbf{i}$ here is a formal symbol. Scaling by complex numbers is given by

$$(a + \mathbf{i}b)\cdot(X \oplus \mathbf{i}Y) = (aX - bY) \oplus \mathbf{i}(bX + aY),$$

and the Lie bracket on $\mathfrak{g}^{\mathbb{C}}$ is obtained by linearly extending the Lie bracket on $\mathfrak{g}$.

Proposition 2.11. *The following maps are complex Lie algebra isomorphisms:*

$$\mathfrak{gl}(n,\mathbb{R})^{\mathbb{C}} \longrightarrow \mathfrak{gl}(n,\mathbb{C}), \qquad X \oplus \mathbf{i}Y \longmapsto X + \mathbf{i}Y, \tag{2.9}$$

$$\mathfrak{sl}(n,\mathbb{R})^{\mathbb{C}} \longrightarrow \mathfrak{sl}(n,\mathbb{C}), \qquad X \oplus \mathbf{i}Y \longmapsto X + \mathbf{i}Y, \tag{2.10}$$

$$\mathfrak{sp}(2n,\mathbb{R})^{\mathbb{C}} \longrightarrow \mathfrak{sp}(2n,\mathbb{C}), \qquad X \oplus \mathbf{i}Y \longmapsto X + \mathbf{i}Y, \tag{2.11}$$

$$\mathfrak{so}(p,q)^{\mathbb{C}} \longrightarrow \mathfrak{aut}(b_{p,q}), \qquad X \oplus \mathbf{i}Y \longmapsto X + \mathbf{i}Y, \tag{2.12}$$

$$\mathfrak{su}(p,q)^{\mathbb{C}} \longrightarrow \mathfrak{sl}(p+q,\mathbb{C}), \qquad X \oplus \mathbf{i}Y \longmapsto X + \mathbf{i}Y, \tag{2.13}$$

$$\mathfrak{gl}(n,\mathbb{H})^{\mathbb{C}} \longrightarrow \mathfrak{gl}(2n,\mathbb{C}), \qquad X \oplus \mathbf{i}Y \longmapsto \Psi(X) + \mathbf{i}\Psi(Y), \tag{2.14}$$

$$\mathfrak{sl}(n,\mathbb{H})^{\mathbb{C}} \longrightarrow \mathfrak{sl}(2n,\mathbb{C}), \qquad X \oplus \mathbf{i}Y \longmapsto \Psi(X) + \mathbf{i}\Psi(Y), \tag{2.15}$$

$$\mathfrak{sp}(p,q)^{\mathbb{C}} \longrightarrow \mathfrak{aut}(\omega'), \qquad X \oplus \mathbf{i}Y \longmapsto \Psi(X) + \mathbf{i}\Psi(Y), \tag{2.16}$$

$$\mathfrak{so}(n,\mathbb{H})^{\mathbb{C}} \longrightarrow \mathfrak{aut}(k), \qquad X \oplus \mathbf{i}Y \longmapsto \Psi(X) + \mathbf{i}\Psi(Y), \tag{2.17}$$

where $b_{p,q} : \mathbb{C}^{p+q} \times \mathbb{C}^{p+q} \to \mathbb{C}$ *is the symmetric bilinear form given by*

$$b_{p,q}(v,w) := v^{\top} I_{p,q} w,$$

while $\omega' : \mathbb{C}^{2(p+q)} \times \mathbb{C}^{2(p+q)} \to \mathbb{C}$ *and* $k : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \to \mathbb{C}$ *are given in Proposition 2.8.*

Remark 2.12. By Proposition 2.4 and Proposition 2.6, it follows that

$$\mathfrak{aut}(b_{p,q}) \cong \mathfrak{so}(p+q,\mathbb{C}), \qquad \mathfrak{aut}(\omega') \cong \mathfrak{sp}(2p+2q,\mathbb{C}), \qquad \mathfrak{aut}(k) \cong \mathfrak{so}(2n,\mathbb{C}),$$

where the isomorphisms here are given by conjugation by some $A \in \mathrm{GL}(m,\mathbb{C})$.

Proof of Proposition 2.11. If $f : \mathfrak{g} \to \mathfrak{h}$ is a real Lie algebra homomorphism, then the map $f^{\mathbb{C}} : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{h}^{\mathbb{C}}$ given by $X \oplus \mathbf{i}Y \mapsto f(X) \oplus \mathbf{i}f(Y)$ is a complex Lie algebra homomorphism. If $\mathfrak{g}$ is already a complex Lie algebra, then the map $\pi : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}$ given by $X \oplus \mathbf{i}Y \mapsto X + \mathbf{i}Y$ is a Lie algebra homomorphism.

Now, (2.9) is nothing more than the composition of the inclusion map : $\iota^{\mathbb{C}} : \mathfrak{gl}(n,\mathbb{R})^{\mathbb{C}} \hookrightarrow \mathfrak{gl}(n,\mathbb{C})^{\mathbb{C}}$ and $\pi : \mathfrak{gl}(n,\mathbb{C})^{\mathbb{C}} \to \mathfrak{gl}(n,\mathbb{C})$. Clearly (2.9) is bijective.

For (2.10), (2.11), (2.12), observe that the image of $\mathfrak{sl}(n,\mathbb{R})^{\mathbb{C}}$, $\mathfrak{sp}(2n,\mathbb{R})^{\mathbb{C}}$, and $\mathfrak{so}(p,q)^{\mathbb{C}}$ are contained in $\mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{sp}(2n,\mathbb{C})$, and $\mathfrak{aut}(b_{p,q})$, respectively, because the latter three are complex vector spaces containing the first three. Surjectivity of (2.10), (2.11), (2.12) follows for dimensional reasons (we can look at the block matrix forms of $\mathfrak{sp}(2n,\mathbb{R})$ and $\mathfrak{so}(p,q)$ to determine their dimensions). The arguments to show (2.15), (2.16), (2.17) are isomorphisms are identical.

The map (2.13) is the restriction of $\pi : \mathfrak{gl}(n,\mathbb{C})^{\mathbb{C}} \to \mathfrak{gl}(n,\mathbb{C})$ to $\mathfrak{su}(p,q)^{\mathbb{C}}$. The image of this restriction is clearly contained in $\mathfrak{sl}(p+q,\mathbb{C})$, and this restriction is injective because $\mathfrak{su}(p,q) \cap \mathbf{i}\mathfrak{su}(p,q) = 0$.

The map (2.14) is the composition of the maps $\Psi^{\mathbb{C}} : \mathfrak{gl}(n,\mathbb{H})^{\mathbb{C}} \hookrightarrow \mathfrak{gl}(2n,\mathbb{C})^{\mathbb{C}}$ and $\pi : \mathfrak{gl}(n,\mathbb{C})^{\mathbb{C}} \to \mathfrak{gl}(n,\mathbb{C})$. Writing (2.14) in block matrices shows that it is injective. Surjectivity follows for dimensional reasons. □

2.3.2 The Killing form

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{K} = \mathbb{R}, \mathbb{C}$. The *Killing form of* $\mathfrak{g}$ is the symmetric bilinear form $B_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ defined by

$$b_{\mathfrak{g}}(X,Y) := \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y),$$

where $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ is the linear map $Y \mapsto [X,Y]$.

Table 2.4 gives the Killing forms of the complex Lie algebras $\mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{so}(n,\mathbb{C})$ and $\mathfrak{sp}(2n,\mathbb{C})$ (see [FH91, Exercise 14.36 on Page 210 and solutions on Page 525]):

$\mathfrak{g}$	$B_{\mathfrak{g}}(X,Y)$
$\mathfrak{sl}(n,\mathbb{C})$	$2n\,\mathrm{tr}(XY)$
$\mathfrak{so}(n,\mathbb{C})$	$(n-2)\,\mathrm{tr}(XY)$
$\mathfrak{sp}(2n,\mathbb{C})$	$(2n+2)\,\mathrm{tr}(XY)$

Table 2.4: Killing forms of complex Lie algebras

Proposition 2.13. *The Killing forms of the real Lie algebras* $\mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{sl}(n,\mathbb{R})$, $\mathfrak{su}(p,q)$, $\mathfrak{sl}(n,\mathbb{H})$, $\mathfrak{so}(n,\mathbb{C})$, $\mathfrak{so}(p,q)$, $\mathfrak{so}(n,\mathbb{H})$, $\mathfrak{sp}(2n,\mathbb{C})$, $\mathfrak{sp}(2n,\mathbb{R})$ *and* $\mathfrak{sp}(p,q)$ *are given in Table 2.5.*

$\mathfrak{g}$	$B_{\mathfrak{g}}(X,Y)$
$\mathfrak{sl}(n,\mathbb{C})$	$4n\operatorname{Re}\operatorname{tr}(XY)$
$\mathfrak{sl}(n,\mathbb{R})$	$2n\operatorname{tr}(XY)$
$\mathfrak{su}(p,q)$	$(2p+2q)\operatorname{tr}(XY)$
$\mathfrak{sl}(n,\mathbb{H})$	$8n\operatorname{Re}\operatorname{tr}(XY)$
$\mathfrak{so}(n,\mathbb{C})$	$(2n-4)\operatorname{Re}\operatorname{tr}(XY)$
$\mathfrak{so}(p,q)$	$(p+q-2)\operatorname{tr}(XY)$
$\mathfrak{so}(n,\mathbb{H})$	$(4n-4)\operatorname{Re}\operatorname{tr}(XY)$
$\mathfrak{sp}(2n,\mathbb{C})$	$(4n+4)\operatorname{Re}\operatorname{tr}(XY)$
$\mathfrak{sp}(2n,\mathbb{R})$	$(2n+2)\operatorname{tr}(XY)$
$\mathfrak{sp}(p,q)$	$(8p+8q+4)\operatorname{Re}\operatorname{tr}(XY)$

Table 2.5: Killing forms of real Lie algebras

Proof. The proposition follows immediately from Proposition 2.11, Table 2.4 and the following two facts. First, if $\mathfrak{g}$ is a complex Lie algebra, and $g^{\mathbb{R}}$ denotes the same Lie algebra considered as a real Lie algebra, then [Kna96, Equation 1.57] tells us that their Killing forms are related via

$$B_{\mathfrak{g}^{\mathbb{R}}} = 2\operatorname{Re} B_{\mathfrak{g}}.$$

Next, $\mathfrak{g}$ is a real Lie algebra and $g^{\mathbb{C}}$ denotes its complexification, then [Kna96, Equation 1.20] tells us that their Killing forms are related via

$$B_{\mathfrak{g}}(X,Y) = B_{g^{\mathbb{C}}}(X\oplus 0, Y\oplus 0).$$

For the Killing forms of $\mathfrak{sl}(n,\mathbb{H})$, $\mathfrak{so}(n,\mathbb{H})$ and $\mathfrak{sp}(p,q)$, also recall that $\operatorname{tr}(\Psi(X)) = 2\operatorname{Re}\operatorname{tr}(X)$, where Ψ is defined in §2.1.4. □

2.3.3 Compact and non-compact type

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{K}=\mathbb{R},\mathbb{C}$. We say that $\mathfrak{g}$ is *simple* if $\mathfrak{g}$ is not abelian and has no non-zero proper ideals. We say that $\mathfrak{g}$ is *semisimple* if $\mathfrak{g}$ is isomorphic to a sum of simple Lie algebras. Clearly every simple Lie algebra is semisimple. A Lie algebra $\mathfrak{g}$ is semisimple if and only if its Killing form $B:\mathfrak{g}\times\mathfrak{g}\to\mathbb{K}$ is non-degenerate (see [Kna96, Theorem 1.42 and Theorem 1.51]).

Now, let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{R}$ with Killing form B. We say that $\mathfrak{g}$ is of *compact type* if B is negative-definite, and $\mathfrak{g}$ is of *non-compact type* if B is not negative-definite.

Proposition 2.14 ([Kna96, §IV.4])**.** *Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{R}$. The following are equivalent:*

(i) *$\mathfrak{g}$ is of compact type.*

(ii) *If G is a connected Lie group whose Lie algebra is $\mathfrak{g}$, then G is compact.*

(iii) *There exists a compact Lie group whose Lie algebra is $\mathfrak{g}$.*

Proof. Clearly (ii) implies (iii), because there exists a connected Lie group whose Lie algebra is $\mathfrak{g}$.

To show that (i) implies (ii), suppose $\mathfrak{g}$ is of compact type, and let G be a connected Lie group whose Lie algebra is $\mathfrak{g}$. Then $-B$ is an $\operatorname{Ad}(G)$-invariant (positive-definite) inner product on $\mathfrak{g} \cong T_eG$, so the corresponding left-invariant Riemannian metric g on G is also bi-invariant [Lee18, Proposition 3.12]. By [Lee18, Problem 7-13], it follows that the Ricci curvature of (G, g) at the identity is $-\frac{1}{4}B > 0$. Since (G, g) is homogeneous, the Bonnet-Myers theorem [Lee18, Theorem 12.24] implies that G is compact.

Finally, to show that (iii) implies (i), let G be a compact Lie group whose Lie algebra is $\mathfrak{g}$. Since G is compact, [Lee18, Corollary 3.15] implies that G admits a bi-invariant Riemannian metric. Let $\langle\cdot,\cdot\rangle$ denote the corresponding $\mathrm{Ad}(G)$-invariant inner product of G. Differentiating at the identity shows that $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ is skew-symmetric with respect to $\langle\cdot,\cdot\rangle$ for each $X \in \mathfrak{g}$. Thus, the eigenvalues of $\mathrm{ad}(X)$ are purely imaginary, so the eigenvalues of $\mathrm{ad}(X)^2$ are real and non-positive. It follows that $B(X, X) = \mathrm{tr}(\mathrm{ad}(X)^2) \leq 0$. Therefore, B is negative-definite: any non-degenerate negative-semidefinite symmetric bilinear form is also negative-definite. □

2.3.4 Classification of simple Lie algebras over $\mathbb{R}$ and $\mathbb{C}$

Proposition 2.15 ([Kna96, Theorem 2.84])**.** *Every simple Lie algebra over $\mathbb{C}$ is isomorphic to one of 5 exceptional Lie algebras, or isomorphic to some $\mathfrak{g}$ listed in Table 2.6.*

$\mathfrak{g}$	Condition	$\dim_{\mathbb{C}} \mathfrak{g}$
$\mathfrak{sl}(n, \mathbb{C})$	$n \geq 2$	$n^2 - 1$
$\mathfrak{so}(n, \mathbb{C})$	$n \geq 3,\ n \neq 4$	$\frac{1}{2}n(n-1)$
$\mathfrak{sp}(2n, \mathbb{C})$	$n \geq 1$	$n(2n+1)$

Table 2.6: Simple Lie algebras over $\mathbb{C}$

Conversely, every $\mathfrak{g}$ listed in Table 2.6 is simple. Moreover, the only isomorphisms among the $\mathfrak{g}$ in Table 2.6 and the 5 exceptional Lie algebras are

$$\mathfrak{sl}(2, \mathbb{C}) \cong \mathfrak{so}(3, \mathbb{C}) \cong \mathfrak{sp}(2, \mathbb{C}), \qquad \mathfrak{so}(5, \mathbb{C}) \cong \mathfrak{sp}(4, \mathbb{C}), \qquad \mathfrak{so}(6, \mathbb{C}) \cong \mathfrak{sl}(4, \mathbb{C}).$$

Remark 2.16. We also have the following low-dimensional isomorphisms:

$$\mathfrak{sl}(1, \mathbb{C}) \cong \mathfrak{so}(1, \mathbb{C}) \cong \{0\}, \qquad \mathfrak{so}(2, \mathbb{C}) \cong \mathbb{C}, \qquad \mathfrak{so}(4, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}).$$

Let $\mathfrak{h}$ be a Lie algebra over $\mathbb{C}$. A *real form* of $\mathfrak{h}$ is a Lie algebra over $\mathbb{R}$ whose complexification is isomorphic to $\mathfrak{h}$. On the other hand, the *realification* of $\mathfrak{h}$ is the Lie algebra $\mathfrak{h}^{\mathbb{R}}$ obtained by considering $\mathfrak{h}$ as a real Lie algebra in the obvious way.

The classification of simple Lie algebras over $\mathbb{R}$ relies on the classification over $\mathbb{C}$. [Kna96, Propositions 6.94 and 6.95] tell us that every simple Lie algebra over $\mathbb{R}$ is either

(i) a real form of a simple Lie algebra over $\mathbb{C}$, or

(ii) a realification of a simple Lie algebra over $\mathbb{C}$.

Conversely, Lie algebra of the form (i) or (ii) is a simple Lie algebra over $\mathbb{R}$.

Proposition 2.17 ([Kna96, Theorem 6.105])**.** *Every simple Lie algebra over $\mathbb{R}$ is isomorphic to one of 22 exceptional Lie algebras, isomorphic to the realification of a complex Lie algebra listed in Table 2.6, or isomorphic to some $\mathfrak{g}$ listed in Tables 2.7 and 2.8.*

Conversely, every $\mathfrak{g}$ listed in Tables 2.6, 2.7 and 2.8 is a simple Lie algebra over $\mathbb{R}$. Moreover, the only isomorphisms among the 22 exceptional Lie algebras and the $\mathfrak{g}$ listed in Tables 2.6, 2.7 and 2.8 are

$\mathfrak{g}$	Condition	$\mathfrak{g}^{\mathbb{C}}$	$\dim\mathfrak{g}$
$\mathfrak{su}(n)$	$n \geq 2$	$\mathfrak{sl}(n,\mathbb{C})$	n^2-1
$\mathfrak{so}(n)$	$n \geq 3,\ n \neq 4$	$\mathfrak{so}(n,\mathbb{C})$	$\frac{1}{2}n(n-1)$
$\mathfrak{sp}(n)$	$n \geq 1$	$\mathfrak{sp}(2n,\mathbb{C})$	$n(2n+1)$

Table 2.7: Real forms of compact type

$\mathfrak{g}$	Condition	$\mathfrak{g}^{\mathbb{C}}$	$\dim\mathfrak{g}$
$\mathfrak{su}(p,q)$	$p \geq q \geq 1$	$\mathfrak{sl}(p+q,\mathbb{C})$	$(p+q)^2-1$
$\mathfrak{sl}(n,\mathbb{R})$	$n \geq 2$	$\mathfrak{sl}(n,\mathbb{C})$	n^2-1
$\mathfrak{sl}(n,\mathbb{H})$	$n \geq 2$	$\mathfrak{sl}(2n,\mathbb{C})$	$4n^2-1$
$\mathfrak{so}(p,q)$	$p \geq q \geq 1,\ p+q \geq 3,\ p+q \neq 4$	$\mathfrak{so}(p+q,\mathbb{C})$	$\frac{1}{2}(p+q)(p+q-1)$
$\mathfrak{so}(n,\mathbb{H})$	$n \geq 3$	$\mathfrak{so}(2n,\mathbb{C})$	$n(2n-1)$
$\mathfrak{sp}(p,q)$	$p \geq q \geq 1$	$\mathfrak{sp}(2(p+q),\mathbb{C})$	$(p+q)(2p+2q-1)$
$\mathfrak{sp}(2n,\mathbb{R})$	$n \geq 1$	$\mathfrak{sp}(2n,\mathbb{C})$	$n(2n+1)$

Table 2.8: Real forms of non-compact type

(i) $\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{so}(3,\mathbb{C}) \cong \mathfrak{sp}(2,\mathbb{C}) \cong \mathfrak{so}(3,1)$; $\mathfrak{su}(2) \cong \mathfrak{sl}(1,\mathbb{H}) \cong \mathfrak{so}(3) \cong \mathfrak{sp}(1)$; $\mathfrak{su}(1,1) \cong \mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{so}(2,1) \cong \mathfrak{sp}(2,\mathbb{R})$.

(ii) $\mathfrak{so}(5,\mathbb{C}) \cong \mathfrak{sp}(4,\mathbb{C})$; $\mathfrak{so}(5) \cong \mathfrak{sp}(2)$; $\mathfrak{so}(4,1) \cong \mathfrak{sp}(1,1)$; $\mathfrak{so}(3,2) \cong \mathfrak{sp}(4,\mathbb{R})$.

(iii) $\mathfrak{sl}(4,\mathbb{C}) \cong \mathfrak{so}(6,\mathbb{C})$; $\mathfrak{su}(4) \cong \mathfrak{so}(6)$; $\mathfrak{su}(3,1) \cong \mathfrak{so}(3,\mathbb{H})$; $\mathfrak{su}(2,2) \cong \mathfrak{so}(4,2)$; $\mathfrak{sl}(4,\mathbb{R}) \cong \mathfrak{so}(3,3)$; $\mathfrak{sl}(2,\mathbb{H}) \cong \mathfrak{so}(5,1)$.

(iv) $\mathfrak{so}(4,\mathbb{H}) \cong \mathfrak{so}(6,2)$.

Ignoring the 22 exceptional simple Lie algebras, the simple Lie algebras of compact type are precisely the $\mathfrak{g}$ listed in Table 2.7, and the simple Lie algebras of non-compact type are precisely the $\mathfrak{g}$ listed in Tables 2.6 and 2.8.

Remark 2.18. We also have the following low-dimensional isomorphisms:

(i) $\mathfrak{sl}(1,\mathbb{C}) \cong \mathfrak{su}(1) \cong \mathfrak{sl}(1,\mathbb{R}) \cong \mathfrak{so}(1,\mathbb{C}) \cong \mathfrak{so}(1) \cong \{0\}$.

(ii) $\mathfrak{so}(2,\mathbb{C}) \cong \mathbb{C}$; $\mathfrak{so}(2) \cong \mathfrak{so}(1,1) \cong \mathfrak{so}(1,\mathbb{H}) \cong \mathbb{R}$.

(iii) $\mathfrak{so}(4,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$; $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$; $\mathfrak{so}(2,2) \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$; $\mathfrak{so}(2,\mathbb{H}) \cong \mathfrak{su}(2) \oplus \mathfrak{sl}(2,\mathbb{R})$.

2.3.5 Cartan decomposition

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{R}$ with Killing form B. A *Cartan involution* of $\mathfrak{g}$ is a Lie algebra automorphism $\theta : \mathfrak{g} \to \mathfrak{g}$ such that $\theta^2 = \mathrm{Id}_{\mathfrak{g}}$ and $B(X,\theta(X)) < 0$ for all non-zero $X \in \mathfrak{g}$. A Cartan involution of $\mathfrak{g}$ always exists, and is unique up to automorphisms of $\mathfrak{g}$ [Kna96, Corollaries 6.18 and 6.19].

On the other hand, a *Cartan decomposition* of $\mathfrak{g}$ is a vector space decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ such that

$$[\mathfrak{k},\mathfrak{k}] \subseteq \mathfrak{k}, \qquad [\mathfrak{k},\mathfrak{p}] \subseteq \mathfrak{p}, \qquad [\mathfrak{p},\mathfrak{p}] \subseteq \mathfrak{k},$$

and the Killing form B is negative-definite on $\mathfrak{k}$ and positive-definite on $\mathfrak{p}$. We have a bijection

$$\Big\{\text{Cartan involutions of } \mathfrak{g}\Big\} \longleftrightarrow \Big\{\text{Cartan decompositions of } \mathfrak{g}\Big\}.$$

If $\theta : \mathfrak{g} \to \mathfrak{g}$ is a Cartan involution, then the corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is given by the ± 1 eigenvalues of θ [Kna96, Page 303]:

$$\mathfrak{k} = \Big\{ X \in \mathfrak{g} : \theta(X) = X \Big\}, \qquad \mathfrak{p} = \Big\{ X \in \mathfrak{g} : \theta(X) = -X \Big\}$$

The purpose of this subsection is to compute Cartan decompositions for the ten infinite families of simple Lie algebras of non-compact type.

Proposition 2.19. *Each Lie algebra $\mathfrak{g}$ listed in Tables 2.6 and 2.8 is closed under conjugate transpose, and a Cartan involution is given by $\theta : X \mapsto -X^*$. The Cartan decomposition corresponding to $X \mapsto -X^*$ is*

$$\mathfrak{k} = \mathfrak{g} \cap \Big\{ \text{Skew-Hermitian matrices} \Big\}, \qquad \mathfrak{p} = \mathfrak{g} \cap \Big\{ \text{Hermitian matrices} \Big\}.$$

Proof. It is easy to see that the Lie algebras in Tables 2.6 and 2.8 are closed under conjugate transpose. Let $\mathfrak{g}$ be one of the Lie algebras listed in these tables. Since $(XY)^* = Y^* X^*$, it follows that $\theta : X \mapsto -X^*$ is a Lie algebra automorphism of $\mathfrak{g}$ such that $\theta^2 = \mathrm{Id}_{\mathfrak{g}}$.

From Table 2.5, we see that the Killing form of $\mathfrak{g}$ is given by $B(X, Y) = \lambda \operatorname{Re} \operatorname{tr}(XY)$ for some $\lambda > 0$. Therefore,

$$B(X, \theta(X)) = -B(X, X^*) = -\lambda \operatorname{tr}(XX^*) < 0$$

for all non-zero $X \in \mathfrak{g}$. $\square$

$\mathfrak{g}$	$\mathfrak{k}$	$\mathfrak{p}$
$\mathfrak{sl}(n, \mathbb{R})$	$\mathfrak{so}(n)$	$\mathrm{Herm}(n, \mathbb{R}) \cap \mathfrak{sl}(n, \mathbb{R})$
$\mathfrak{sl}(n, \mathbb{C})$	$\mathfrak{su}(n)$	$\mathrm{Herm}(n, \mathbb{C}) \cap \mathfrak{sl}(n, \mathbb{C})$
$\mathfrak{sl}(n, \mathbb{H})$	$\mathfrak{sp}(n)$	$\mathrm{Herm}(n, \mathbb{H}) \cap \mathfrak{sl}(n, \mathbb{H})$
$\mathfrak{so}(p, q)$	$\left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : A \in \mathfrak{so}(p),\ D \in \mathfrak{so}(q) \right\}$	$\left\{ \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} : B \in \mathrm{M}_{p \times q}(\mathbb{R}) \right\}$
$\mathfrak{su}(p, q)$	$\left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : \begin{matrix} A \in \mathfrak{su}(p),\ D \in \mathfrak{su}(q) \\ \operatorname{tr}(A) + \operatorname{tr}(D) = 0 \end{matrix} \right\}$	$\left\{ \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} : B \in \mathrm{M}_{p \times q}(\mathbb{C}) \right\}$
$\mathfrak{sp}(p, q)$	$\left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : A \in \mathfrak{sp}(p),\ D \in \mathfrak{sp}(q) \right\}$	$\left\{ \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} : B \in \mathrm{M}_{p \times q}(\mathbb{H}) \right\}$
$\mathfrak{sp}(2n, \mathbb{R})$	$\left\{ \begin{pmatrix} A & -C \\ C & A \end{pmatrix} : \begin{matrix} A \in \mathfrak{so}(n) \\ C \in \mathrm{Sym}(n, \mathbb{R}) \end{matrix} \right\}$	$\left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} : A, B \in \mathrm{Sym}(n, \mathbb{R}) \right\}$
$\mathfrak{sp}(2n, \mathbb{C})$	$\left\{ \begin{pmatrix} A & -\overline{C} \\ C & \overline{A} \end{pmatrix} : \begin{matrix} A \in \mathfrak{u}(n) \\ C \in \mathrm{Sym}(n, \mathbb{C}) \end{matrix} \right\}$	$\left\{ \begin{pmatrix} A & B \\ \overline{B} & -\overline{A} \end{pmatrix} : \begin{matrix} A \in \mathrm{Herm}(n, \mathbb{C}) \\ B \in \mathrm{Sym}(n, \mathbb{C}) \end{matrix} \right\}$
$\mathfrak{so}(n, \mathbb{C})$	$\mathfrak{so}(n)$	$\mathfrak{iso}(n)$
$\mathfrak{so}(n, \mathbb{H})$	$\mathfrak{u}(n)$	$\mathfrak{jso}(n, \mathbb{C})$

Table 2.9: Cartan decompositions $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for all the simple real Lie algebras of non-compact type $\mathfrak{g}$ listed in Tables 2.6 and 2.8. Note that $\mathfrak{k} = \Phi(\mathfrak{u}(n))$ when $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{R})$, and $\mathfrak{k} = \Psi(\mathfrak{sp}(n))$ when $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$, where Φ and Ψ are the embeddings defined in §2.1.4.

2.4 Automorphism groups of simple Lie algebras over $\mathbb{R}$

The aim of this section is to prove the following theorem:

Theorem 2.20. *Let* $\mathfrak{g}$ *be a simple real Lie algebra listed in Tables 2.6 and 2.8, with* $\mathfrak{g} \neq \mathfrak{so}(4,4), \mathfrak{so}(8,\mathbb{C})$*. The automorphism group* $\mathrm{Aut}(\mathfrak{g})$ *of* $\mathfrak{g}$ *is given by*

$$\mathrm{Aut}(\mathfrak{g}) = \mathrm{Ad}(G') \cdot \Gamma',$$

where G' *is the matrix Lie group with Lie algebra* $\mathfrak{g}$ *listed in Table 2.10,* Γ' *is the finite subgroup of* $\mathrm{Aut}(\mathfrak{g})$ *listed in Table 2.10, and* $\mathrm{Ad} : G' \to \mathrm{Aut}(\mathfrak{g})$ *denotes the adjoint representation of* G'*, which is given by* $\mathrm{Ad}(A)(X) = AXA^{-1}$.

$\mathfrak{g}$	G'	G	$\ker(\mathrm{Ad})$	Γ'
$\mathfrak{sl}(n,\mathbb{R})$	$\mathrm{SL}_{\pm}(n,\mathbb{R})$	$\mathrm{SL}(n,\mathbb{R})$	$\pm\,\mathrm{I}_n$	$X \mapsto -X^{\top}$ when $n \geq 3$
$\mathfrak{sl}(n,\mathbb{C})$	$\mathrm{SL}(n,\mathbb{C})$	$\mathrm{SL}(n,\mathbb{C})$	$\lambda\,\mathrm{I}_n : \lambda^n = 1$	$X \mapsto -X^{*}$ when $n \geq 3$, $X \mapsto \overline{X}$
$\mathfrak{sl}(n,\mathbb{H})$	$\mathrm{SL}(n,\mathbb{H})$	$\mathrm{SL}(n,\mathbb{H})$	$\pm\,\mathrm{I}_n$	$X \mapsto -X^{*}$
$\mathfrak{so}(p,q)$	$\mathrm{O}(p,q)$	$\mathrm{O}(p,q)^0$	$\pm\,\mathrm{I}_{p+q}$	$\mathrm{Ad}\left(\begin{smallmatrix} 0 & \mathrm{I}_p \\ \mathrm{I}_p & 0 \end{smallmatrix}\right)$ when $p = q$
$\mathfrak{su}(p,q)$	$\mathrm{SU}(p,q)$	$\mathrm{SU}(p,q)$	$\lambda I_{p+q} : \lambda^n = 1$	$\mathrm{Ad}\left(\begin{smallmatrix} 0 & \mathrm{I}_p \\ \mathrm{I}_p & 0 \end{smallmatrix}\right)$ when $p = q \geq 2$, $X \mapsto \overline{X}$
$\mathfrak{sp}(p,q)$	$\mathrm{Sp}(p,q)$	$\mathrm{Sp}(p,q)$	$\pm\,\mathrm{I}_{p+q}$	$\mathrm{Ad}\left(\begin{smallmatrix} 0 & \mathrm{I}_p \\ \mathrm{I}_p & 0 \end{smallmatrix}\right)$ when $p = q$
$\mathfrak{sp}(2n,\mathbb{R})$	$\mathrm{Sp}(2n,\mathbb{R})$	$\mathrm{Sp}(2n,\mathbb{R})$	$\pm\,\mathrm{I}_{2n}$	$\mathrm{Ad}\left(\begin{smallmatrix} \mathrm{I}_n & 0 \\ 0 & -\mathrm{I}_n \end{smallmatrix}\right)$
$\mathfrak{sp}(2n,\mathbb{C})$	$\mathrm{Sp}(2n,\mathbb{C})$	$\mathrm{Sp}(2n,\mathbb{C})$	$\pm\,\mathrm{I}_{2n}$	$X \mapsto \overline{X}$
$\mathfrak{so}(n,\mathbb{C})$	$\mathrm{O}(n,\mathbb{C})$	$\mathrm{SO}(n,\mathbb{C})$	$\pm\,\mathrm{I}_n$	$X \mapsto \overline{X}$
$\mathfrak{so}(n,\mathbb{H})$	$\mathrm{SO}(n,\mathbb{H})$	$\mathrm{SO}(n,\mathbb{H})$	$\pm\,\mathrm{I}_n$	$\mathrm{Ad}(\mathbf{j}\,\mathrm{I}_n)$

Table 2.10: Data for Theorem 2.20.

Proof of Theorem 2.20. Here is the strategy:

(i) The identity component $\mathrm{Aut}(\mathfrak{g})^0$ of $\mathrm{Aut}(\mathfrak{g})$ is equal to $\mathrm{Ad}(G)$, where G is the identity component of G' [Kna96, Proposition 1.98]. The group G is listed in Table 2.10.

(ii) The number of connected components of $\mathrm{Aut}(\mathfrak{g})$ is given by [Gun10, Corollary 2.15, Corollary 2.22, Table 1]. We have summarised the results in Table 2.11.

Therefore, it suffices to show that the number of connected components of $\mathrm{Ad}(G') \cdot \Gamma'$ and $\mathrm{Aut}(\mathfrak{g})$ are equal. Part (iii) of the following lemma gives a formula for the number of connected components of $\mathrm{Ad}(G') \cdot \Gamma'$. Assuming Lemma 2.21 to be true, Theorem 2.20 follows immediately from Table 2.12 and Part (iii) of Lemma 2.21. ☐

Lemma 2.21. *Let* $\mathfrak{g}$*,* G'*,* G*,* Γ' *be as given in Table 2.10. Then*

(i) *The kernel of the adjoint representation* $\mathrm{Ad} : G' \to \mathrm{Aut}(\mathfrak{g})$ *is given by the fourth column of Table 2.10.*

$\mathfrak{g}$	Condition	$\mathrm{Aut}(\mathfrak{g})/\mathrm{Aut}(\mathfrak{g})^0$	$\lvert\mathrm{Aut}(\mathfrak{g})/\mathrm{Aut}(\mathfrak{g})^0\rvert$
$\mathfrak{sl}(2,\mathbb{R})$	–	$\mathbb{Z}_2$	2
$\mathfrak{sl}(n,\mathbb{R})$	$n \geq 4$ even	$\mathbb{Z}_2 \times \mathbb{Z}_2$	4
$\mathfrak{sl}(n,\mathbb{R})$	$n \geq 3$ odd	$\mathbb{Z}_2$	2
$\mathfrak{sl}(2,\mathbb{C})$	–	$\mathbb{Z}_2$	2
$\mathfrak{sl}(n,\mathbb{C})$	$n \geq 3$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	2
$\mathfrak{sl}(n,\mathbb{H})$	$n \geq 2$	$\mathbb{Z}_2$	2
$\mathfrak{so}(p,q)$	$p,q \geq 1,\ p \neq q$, p or q is odd	$\mathbb{Z}_2$	2
$\mathfrak{so}(p,q)$	$p,q \geq 1,\ p \neq q$, p and q are even	$\mathbb{Z}_2 \times \mathbb{Z}_2$	4
$\mathfrak{so}(4,4)$	–	S_4	24
$\mathfrak{so}(p,p)$	$p \geq 6$ even	D_4	8
$\mathfrak{so}(p,p)$	$p \geq 3$ odd	$\mathbb{Z}_2 \times \mathbb{Z}_2$	4
$\mathfrak{su}(p,q)$	$p,q \geq 1,\ p \neq q$	$\mathbb{Z}_2$	2
$\mathfrak{su}(1,1)$	–	$\mathbb{Z}_2$	2
$\mathfrak{su}(p,p)$	$p \geq 2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	4
$\mathfrak{sp}(p,q)$	$p,q \geq 1,\ p \neq q$	1	1
$\mathfrak{sp}(p,p)$	$p \geq 1$	$\mathbb{Z}_2$	2
$\mathfrak{so}(n,\mathbb{C})$	$n \geq 3$ odd	$\mathbb{Z}_2$	2
$\mathfrak{so}(8,\mathbb{C})$	–	$S_3 \times \mathbb{Z}_2$	12
$\mathfrak{so}(n,\mathbb{C})$	$n \geq 6$ even, $n \geq 8$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	4
$\mathfrak{so}(n,\mathbb{H})$	$n \geq 5$	$\mathbb{Z}_2$	2

Table 2.11: Groups of connected components of automorphism groups of (non-exceptional) simple Lie algebras of non-compact type

(ii) $\mathrm{Ad}(G') \cap \Gamma'$ *is trivial.*

(iii) *The number of connected components of* $\mathrm{Ad}(G') \cdot \Gamma'$ *is*

$$|G'/G| \cdot \frac{|\ker(\mathrm{Ad}) \cap G|}{|\ker(\mathrm{Ad})|} \cdot |\Gamma'|.$$

Proof of Lemma 2.21 Part (i). We want to show that the fourth column of Table 2.10 indeed gives the kernel of $\mathrm{Ad} : G' \to \mathrm{Aut}(\mathfrak{g})$. It is easy to see that the matrices listed in Table 2.10 belong to $\ker(\mathrm{Ad})$. It remains to show the other inclusion. The following facts are useful:

(a) If $A \in \mathrm{M}_n(\mathbb{K})$ commutes with all $X \in \mathrm{M}_n(\mathbb{K})$, then $A = \lambda\,\mathrm{I}_n$, where $\lambda \in \mathbb{R}$ for $\mathbb{K} = \mathbb{R}, \mathbb{H}$, and $\lambda \in \mathbb{C}$ for $\mathbb{K} = \mathbb{C}$ [EH18, Exercise 3.16, Solutions Page 281].

(b) The kernel of $\mathrm{Ad}\,|_G : G \to \mathrm{Aut}(\mathfrak{g})$ is $Z(G)$, the center of G [Lee13, Problem 20-20].

First, consider $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$. Let $A \in \mathrm{SL}_{\pm}(n,\mathbb{R})$, and suppose $A \in \ker(\mathrm{Ad})$. We want to show that $A = \pm\,\mathrm{I}_n$. Now, A commutes with all $X \in \mathfrak{sl}(n,\mathbb{R})$. Since A also commutes with I_n, it follows that A commutes with every element of $\mathrm{M}_n(\mathbb{R})$. Therefore, by (a), we

$\mathfrak{g}$	Condition	$\lvert G'/G\rvert$	$\lvert\ker(\mathrm{Ad})\rvert$	$\lvert\ker(\mathrm{Ad})\cap G\rvert$	$\lvert\Gamma'\rvert$
$\mathfrak{sl}(2,\mathbb{R})$	–	2	2	2	1
$\mathfrak{sl}(n,\mathbb{R})$	$n \geq 4$ even	2	2	2	2
$\mathfrak{sl}(n,\mathbb{R})$	$n \geq 3$ odd	2	2	1	2
$\mathfrak{sl}(2,\mathbb{C})$	–	1	2	2	2
$\mathfrak{sl}(n,\mathbb{C})$	$n \geq 3$	1	n	n	4
$\mathfrak{sl}(n,\mathbb{H})$	$n \geq 2$	1	2	2	2
$\mathfrak{so}(p,q)$	$p,q \geq 1,\ p \neq q,$ p or q is odd	4	2	1	1
$\mathfrak{so}(p,q)$	$p,q \geq 1,\ p \neq q,$ p and q are even	4	2	2	1
$\mathfrak{so}(p,p)$	$p \geq 6$ even	4	2	2	2
$\mathfrak{so}(p,p)$	$p \geq 3$ odd	4	2	1	2
$\mathfrak{su}(p,q)$	$p,q \geq 1,\ p \neq q$	1	$p+q$	$p+q$	2
$\mathfrak{su}(1,1)$	–	1	2	2	2
$\mathfrak{su}(p,p)$	$p \geq 2$	1	$2p$	$2p$	4
$\mathfrak{sp}(p,q)$	$p,q \geq 1,\ p \neq q$	1	2	2	1
$\mathfrak{sp}(p,p)$	$p \geq 1$	1	2	2	2
$\mathfrak{so}(n,\mathbb{C})$	$n \geq 3$ odd	2	2	1	2
$\mathfrak{so}(n,\mathbb{C})$	$n \geq 6$ even, $n \geq 8$	2	2	2	2
$\mathfrak{so}(n,\mathbb{H})$	$n \geq 5$	1	2	2	2

Table 2.12: Numerical data for Theorem 2.20.

know $A = \lambda\,\mathrm{I}_n$ for some $\lambda \in \mathbb{R}$. Since $\det(A) = \pm 1$, it follows that $\lambda = \pm 1$. The cases $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$ and $\mathfrak{sl}(n,\mathbb{H})$ are analogous.

Next, consider $\mathfrak{g} = \mathfrak{so}(p,q)$. Let $L \in \mathrm{O}(p,q)$, and write L in block matrix form, $L = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. We know that $L^{-1} = \begin{pmatrix} A^\top & -C^\top \\ -B^\top & D^\top \end{pmatrix}$, and A, D are invertible matrices. Now, suppose $L \in \ker(\mathrm{Ad})$. We want to show that $L = \pm\,\mathrm{I}_{p+q}$. By writing out the equation $LXL^{-1} = X$ for $X \in \mathfrak{so}(p,q)$ in block matrix form, it is straightforward to see that $B = 0$, $C = 0$, $A \in \mathrm{O}(p)$, $D \in \mathrm{O}(q)$, and $AY = YD$ for all $Y \in \mathrm{M}_{p\times q}(\mathbb{R})$. By choosing $Y = E_{ij}$, it follows that $L = \lambda\,\mathrm{I}_{p+q}$ for some $\lambda \in \mathbb{R}$. The condition $L \in \mathrm{O}(p,q)$ implies that $\lambda = \pm 1$. The cases $\mathfrak{g} = \mathfrak{su}(p,q)$ and $\mathfrak{sp}(p,q)$ are analogous.

Consider $\mathfrak{g} = \mathfrak{sp}(2n,\mathbb{K})$ for $\mathbb{K} = \mathbb{R},\mathbb{C}$. Since $G' = G = \mathrm{Sp}(2n,\mathbb{K})$, (b) tells us that $\ker(\mathrm{Ad}) = Z(G)$. It is well known that the center of $\mathrm{Sp}(2n,\mathbb{K})$ is $\pm\,\mathrm{I}_{2n}$ (for example, see [O'M78, Chapter 3.2]).

Consider $\mathfrak{g} = \mathfrak{so}(n,\mathbb{C})$. Let $A \in \mathrm{O}(n,\mathbb{C})$, and suppose $A \in \ker(\mathrm{Ad})$. The standard representation $\mathbb{C}^n$ of $\mathfrak{so}(n,\mathbb{C})$ is irreducible [FH91, Theorems 19.2 and 19.14]. The map $\mathbb{C}^n \to \mathbb{C}^n$ given by $v \mapsto Av$ is an intertwiner, because $AX = XA$ for all $X \in \mathfrak{so}(n,\mathbb{C})$. Therefore, $A = \lambda\,\mathrm{I}_n$ for some $\lambda \in \mathbb{C}$. Then condition $A \in \mathrm{O}(n,\mathbb{C})$ implies $\lambda = \pm 1$.

Finally, consider $\mathfrak{g} = \mathfrak{so}(n,\mathbb{H})$. Since $G' = G = \mathrm{SO}(n,\mathbb{H})$, (b) tells us that $\ker(\mathrm{Ad}) = Z(G)$. Let $A \in \ker(\mathrm{Ad})$. Then [Kna96, Theorem 6.31 Part (e)] tells us that $A \in \mathrm{U}(n)$. In particular, A belongs to the center of $\mathrm{U}(n)$. By Schur's lemma applied to the standard representation of $\mathrm{U}(n)$, it follows that $A = \lambda\,\mathrm{I}_n$ for some $\lambda \in \S^1 \subseteq \mathbb{C}$. The condition $A \in \mathrm{SO}(n,\mathbb{H})$ implies that λ is real, so $\lambda = \pm 1$. ☐

Proof of Lemma 2.21 Part (ii). We want to show that, for each $\mathfrak{g}$, the intersection of

$\mathrm{Ad}(G')$ and Γ' is trivial. It suffices to show that each element of Γ' listed in Table 2.10 has a property that every element of $\mathrm{Ad}(G')$ does not have. In what follows, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ denote the Cartan decomposition given in Table 2.9.

First, consider $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{K})$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, where $n \geq 3$. Consider the Cartan involution $\theta : X \mapsto -X^*$. For each $A \in G'$ and $X \in \mathrm{Herm}(n, \mathbb{K})_0$, the eigenvalues of X and $\mathrm{Ad}(X) = AXA^{-1}$ are the same. However, $X = \mathrm{diag}(2, -1, -1)$ and $\theta(X) = \mathrm{diag}(-2, 1, 1)$ have different eigenvalues. For $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ where $n \geq 3$, an identical argument holds for $X \mapsto -X^\top$. Now, for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{H})$, consider the following elements of $\mathrm{Herm}(2, \mathbb{H})_0$:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & \mathbf{i} \\ -\mathbf{i} & 0 \end{pmatrix}, \qquad Z = XY = \begin{pmatrix} -\mathbf{i} & 0 \\ 0 & \mathbf{i} \end{pmatrix}.$$

For the sake of contradiction, suppose $\theta = \mathrm{Ad}(A)$ for some $A \in \mathrm{SL}(2, \mathbb{H})$. Then

$$-Z = \theta Z = \mathrm{Ad}(A)Z = (\mathrm{Ad}(A)X)(\mathrm{Ad}(A)Y) = (\theta X)(\theta Y) = (-X)(-Y) = Z.$$

Next, consider $\mathfrak{g} = \mathfrak{so}(p, p), \mathfrak{su}(p, p)$ and $\mathfrak{sp}(p, p)$. In this paragraph and the next, we identify

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & Y \\ Y^* & 0 \end{pmatrix} : Y \in \mathrm{M}_p(\mathbb{K}) \right\} \cong \mathrm{M}_p(\mathbb{K})$$

in the obvious manner. Let $\Omega := \left(\begin{smallmatrix} 0 & \mathrm{I}_p \\ \mathrm{I}_p & 0 \end{smallmatrix}\right)$. In block matrix form, we find that

$$\mathrm{Ad}(\Omega) \begin{pmatrix} X & Y \\ Y^* & W \end{pmatrix} = \begin{pmatrix} W & Y^* \\ Y & X \end{pmatrix}.$$

In particular, $\mathrm{Ad}(\Omega)$ preserves $\mathfrak{k}$ and $\mathfrak{p}$, and its restriction to $\mathfrak{p} \cong \mathrm{M}_p(\mathbb{K})$ is given by $Y \mapsto Y^*$. Observe that this restriction is not a ring homomorphism of $\mathrm{M}_p(\mathbb{K})$ when $p \geq 2$, or when $p = 1$ and $\mathbb{K} = \mathbb{H}$. Now, for the sake of contradiction, suppose $\mathrm{Ad}(\Omega) = \mathrm{Ad}(L)$ for some $L \in G'$. Write L in block matrix form, $L = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. Since $\mathrm{Ad}(L)$ preserves $\mathfrak{k}$ and $\mathfrak{p}$, it follows that $B = C = 0$, A, D are unitary, and the restriction of $\mathrm{Ad}(L)$ to $\mathfrak{p} \cong \mathrm{M}_p(\mathbb{K})$ is given by $Y \mapsto AYD^{-1}$. Now, $\mathrm{I}_p = \mathrm{Ad}(\Omega)\,\mathrm{I}_p = \mathrm{Ad}(L)\,\mathrm{I}_p = AD^{-1}$, so $A = D$. Therefore, $\mathrm{Ad}(L)|_{\mathfrak{p}} : Y \mapsto AYA^{-1}$ is always a ring homomorphism of $\mathrm{M}_p(\mathbb{K})$, so we have reached a contradiction. For $\mathfrak{g} = \mathfrak{su}(p, p)$ where $p \geq 2$, an identical argument shows that the composition of $\mathrm{Ad}(\Omega)$ and $X \mapsto \overline{X}$ does not belong to $\mathrm{Ad}(\mathrm{SU}(p, p))$.

For $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C}), \mathfrak{so}(n, \mathbb{C})$, and $\mathfrak{sp}(2n, \mathbb{C})$, the map $X \mapsto \overline{X}$ is not $\mathbb{C}$-linear, but $\mathrm{Ad}(A) : \mathfrak{g} \to \mathfrak{g}$ is $\mathbb{C}$-linear for any $A \in G'$. For $\mathfrak{g} = \mathfrak{su}(p, q)$, the map $X \mapsto \overline{X}$ preserves $\mathfrak{k}$ and $\mathfrak{p}$, and is not $\mathbb{C}$-linear on $\mathfrak{p} \cong \mathrm{M}_{p \times q}(\mathbb{C})$. As in the previous paragraph, if $L = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ belongs to $\mathrm{SU}(p, q)$ and $\mathrm{Ad}(L)$ preserves $\mathfrak{k}$ and $\mathfrak{p}$, then the restriction of $\mathrm{ad}(L)$ to $\mathfrak{p} \cong \mathrm{M}_{p \times q}(\mathbb{C})$ is given by $Y \mapsto AYD^{-1}$, which is $\mathbb{C}$-linear.

For the remaining two cases, $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{R})$ and $\mathfrak{g} = \mathfrak{so}(n, \mathbb{H})$, we use the following fact. Let K denote the connected Lie subgroup of G whose Lie algebra is $\mathfrak{k}$. If $A \in G$ has finite order and $\mathrm{Ad}(A)\mathfrak{k} \subseteq \mathfrak{k}$, then A belongs to K. To see this, first observe that G is a connected, so [Kna96, Theorem 6.31 Part (g)] tells us that K is a maximal compact subgroup. Now, since $\mathrm{Ad}(A)\mathfrak{k} \subseteq \mathfrak{k}$, we know that A normalises K, so $K \cdot \langle A \rangle$ is a compact subgroup of G containing K. Thus, A belongs to K.

Consider $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{R})$. We identify

$$\mathfrak{p} = \left\{ \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} : X, Y \in \mathrm{Sym}(n, \mathbb{R}) \right\} \cong \mathrm{Sym}(n, \mathbb{C}) \quad \text{via} \quad \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \leftrightarrow X - \mathbf{i}Y.$$

Observe that

$$\mathrm{Ad}(\mathrm{I}_{n,n})\begin{pmatrix} X & Y \\ Z & -X^\top \end{pmatrix} = \begin{pmatrix} X & -Y \\ -Z & -X^\top \end{pmatrix},$$

so $\mathrm{Ad}(\mathrm{I}_{n,n})$ preserves $\mathfrak{k}$ and $\mathfrak{p}$. Moreover, the restriction of $\mathrm{Ad}(\mathrm{I}_{n,n})$ to $\mathfrak{p} \cong \mathrm{Sym}(n, \mathbb{C})$ is given by $Z \mapsto \overline{Z}$, which is not $\mathbb{C}$-linear. For the sake of contradiction, suppose $L \in \mathrm{Sp}(2n, \mathbb{R})$ and $\mathrm{Ad}(\mathrm{I}_{n,n}) = \mathrm{Ad}(L)$. Since $\ker(\mathrm{Ad})$ is finite, it follows that L has finite order, so the previous paragraph implies that $L \in \Phi(\mathrm{U}(n))$, where Φ is the embedding defined in §2.1.4. Thus, we can write $L = \left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right)$, where $A + \mathbf{i}B \in \mathrm{U}(n)$. Since $L \in \mathrm{Sp}(2n, \mathbb{R})$, we know that $L^{-1} = \left(\begin{smallmatrix} A^\top & -B^\top \\ B^\top & A^\top \end{smallmatrix}\right)$. Moreover, the restriction of $\mathrm{Ad}(L)$ to $\mathfrak{p} \cong \mathrm{Sym}(n, \mathbb{C})$ is given by $Z \mapsto (A + \mathbf{i}B)Z(A^\top + \mathbf{i}B^\top)$, which is $\mathbb{C}$-linear. Thus, we have reached a contradiction.

Finally, consider $\mathfrak{g} = \mathfrak{so}(n, \mathbb{H})$. Recall that elements of $\mathfrak{so}(n, \mathbb{H})$ are of the form $X + \mathbf{j}Y$, where $X \in \mathfrak{u}(n)$ and $Y \in \mathfrak{so}(n, \mathbb{C})$. Observe that $\mathrm{Ad}(\mathbf{j}\,\mathrm{I}_n)(X + \mathbf{j}Y) = \overline{X} + \mathbf{j}\overline{Y}$, so $\mathrm{Ad}(\mathbf{j}\,\mathrm{I}_n)$ preserves $\mathfrak{k}$ and $\mathfrak{p}$. Moreover, the restriction to $\mathfrak{p} \cong \mathfrak{so}(n, \mathbb{C})$ is given by $Y \mapsto \overline{Y}$, which is not $\mathbb{C}$-linear. For the sake of contradiction, suppose $A \in \mathrm{SO}(n, \mathbb{H})$, and $\mathrm{Ad}(A) = \mathrm{Ad}(\mathbf{j}\,\mathrm{I}_n)$. Then $A \in \mathrm{U}(n)$, and the restriction of $\mathrm{Ad}(A)$ to $\mathfrak{p} \cong \mathfrak{so}(n, \mathbb{C})$ is $Y \mapsto \overline{A}YA^{-1}$, which is $\mathbb{C}$-linear. □

Proof of Lemma 2.21 Part (iii). We want to show that

$$|(\mathrm{Ad}(G') \cdot \Gamma)/G| = |G'/G| \cdot \frac{|\ker(\mathrm{Ad}) \cap G|}{|\ker(\mathrm{Ad})|} \cdot |\Gamma'|. \tag{2.18}$$

First, let $\overline{\mathrm{Ad}} : G'/G \to \mathrm{Aut}(\mathfrak{g})/\mathrm{Aut}(\mathfrak{g})^0$ denote the induced group homomorphism on the connected components. Recall that this is the unique homomorphism such that the following diagram commutes:

$$\begin{tikzcd}
G' \arrow[r, "\mathrm{Ad}"] \arrow[d] & \mathrm{Aut}(\mathfrak{g}) \arrow[d] \\
G'/G \arrow[r, "\overline{\mathrm{Ad}}"'] & \mathrm{Aut}(\mathfrak{g})/\mathrm{Aut}(\mathfrak{g})^0
\end{tikzcd}$$

The vertical arrows denote the natural projection maps. The image of $\overline{\mathrm{Ad}}$ is $\mathrm{Ad}(G')/\mathrm{Aut}(\mathfrak{g})^0$. Thus, the first isomorphism theorem tells us

$$|\mathrm{Ad}(G')/\mathrm{Aut}(\mathfrak{g})^0| = |G'/G| \cdot \frac{1}{|\ker(\overline{\mathrm{Ad}})|}.$$

Let us compute $\ker(\overline{\mathrm{Ad}})$. Let $h : \ker(\mathrm{Ad}) \to G/G'$ denote the composition of maps $\ker(\mathrm{Ad}) \hookrightarrow G \to G/G'$, where the second arrow is the natural projection map. We find that $\ker(h) = \ker(\mathrm{Ad}) \cap G$, and the image of h is $\ker(\overline{\mathrm{Ad}})$. Hence, the first isomorphism theorem tells us the number of connected components of $\mathrm{Ad}(G')$ is

$$|\mathrm{Ad}(G')/\mathrm{Aut}(\mathfrak{g})^0| = |G'/G| \cdot \frac{|\ker(\mathrm{Ad}) \cap G|}{|\ker(\mathrm{Ad})|}. \tag{2.19}$$

Let $\overline{\mathrm{Ad}(G)}$ and $\overline{\Gamma}$ denote the images of $\mathrm{Ad}(G)$ and Γ under the natural projection map $\mathrm{Aut}(\mathfrak{g}) \to \mathrm{Aut}(\mathfrak{g})/\mathrm{Aut}(\mathfrak{g})^0$, respectively. It is easy to see that intersection of $\overline{\mathrm{Ad}(G)}$ and $\overline{\Gamma}$ is trivial, and $|\Gamma| = |\overline{\Gamma}|$. Therefore, we can write

$$|(\mathrm{Ad}(G') \cdot \Gamma)/G| = |\overline{\mathrm{Ad}(G)} \cdot \overline{\Gamma}| = |\overline{\mathrm{Ad}(G)}| \cdot |\overline{\Gamma}| = |\mathrm{Ad}(G')/\mathrm{Aut}(\mathfrak{g})^0| \cdot |\Gamma|. \tag{2.20}$$

Combining Equations 2.19 and 2.20 gives Equation 2.18. □

Chapter 3

Symmetric spaces

Part of the main theorem is proven in §2.4: we compute the automorphism groups of the ten infinite families of simple Lie algebras of non-compact type. The aim of this chapter is to finish the proof of the main theorem: that is, we construct the corresponding symmetric spaces, and compute their isometry groups. This is done in §§3.3, 3.4, 3.5, 3.6, 3.7, and 3.8. To begin the chapter, we recall some important facts about irreducible symmetric spaces of non-compact type in §§3.1 and 3.2.

Throughout this chapter, the reader is assumed to be familiar with smooth manifolds and Riemannian geometry. Some approachable references for these topics are [Lee13], [Tu11], [Lee18], or [Tu17].

3.1 Preliminaries

In this section, we describe how the isometry group of a Riemannian manifold is a Lie group (§3.1.1), then explain what an irreducible symmetric space is (§§3.1.2 and 3.1.3).

3.1.1 Isometry groups

Let (M, g) be a connected Riemannian manifold. Recall that the *isometry group* $\operatorname{Isom}(M, g)$ is the group of all isometries of (M, g). In this subsection, we equip $\operatorname{Isom}(M, g)$ with a smooth manifold structure making it a Lie group.

First, equip $\operatorname{Isom}(M, g)$ with the *compact-open topology*: by definition, this is the topology generated by subsets of the form

$$W(C, U) := \Big\{ f \in \operatorname{Isom}(M, g) : f(C) \subseteq U \Big\},$$

where $C \subseteq M$ is compact, and $U \subseteq M$ is open.

The Myers-Steenrod theorem [Kob95, Chapter II Theorem 1.2] tells us the following. Equipped with the compact-open topology, $\operatorname{Isom}(M, g)$ becomes a topological manifold. Moreover, there exists a unique smooth structure on $\operatorname{Isom}(M, g)$ which makes $\operatorname{Isom}(M, g)$ a Lie group, i.e. multiplication and inversion are smooth. With respect to this smooth structure, the natural action of $\operatorname{Isom}(M, g)$ on M is smooth.

Remark 3.1. If H is a Lie group acting smoothly and isometrically on (M, g), then the induced group homomorphism $H \to \operatorname{Isom}(M, g)$ is smooth: the compact-open topology implies this map is continuous [Mun00, Theorem 46.11], and any continuous homomorphism between Lie groups is smooth [Lee13, Problem 20-11(b)].

3.1.2 Holonomy and irreducibility

Throughout this subsection, let (M, g) be a simply-connected and complete Riemannian manifold.

For each $p \in M$, the *holonomy group* $\mathrm{Hol}_p(M, g)$ is defined as follows. First, a *piecewise-smooth loop* at p is a piecewise-smooth curve $\gamma : [0, 1] \to M$ such that $\gamma(0) = \gamma(1) = p$. For each piecewise-smooth loop γ at p, let $P^\gamma : T_pM \to T_pM$ denote the parallel transport map along γ. We define

$$\mathrm{Hol}_p(M, g) := \Big\{ P^\gamma : \gamma \text{ is a piecewise-smooth loop at } p \Big\}.$$

Since (M, g) is simply-connected, it follows that $\mathrm{Hol}_p(M, g)$ is a connected closed compact subgroup of $\mathrm{SO}(T_pM, g_p)$ [BL52] [Bes08, Chapter 10]. The *holonomy representation* is the natural inclusion map

$$\mathrm{Hol}_p(M, g) \hookrightarrow \mathrm{GL}(T_pM).$$

The holonomy group and the holonomy representation do not depend on the point p in the following sense: if $q \in M$ is another point, then there is a Lie group isomorphism $\mathrm{GL}(T_pM) \cong \mathrm{GL}(T_qM)$ sending $\mathrm{Hol}_p(M, g)$ to $\mathrm{Hol}_q(M, g)$ [Lee18, Problem 5-23(c)].

We say that a (simply-connected and complete) Riemannian manifold is *irreducible* if it is not the product of two Riemannian manifolds of smaller dimension. The De Rham decomposition theorem [KN96] tells us that (M, g) is isometric to a product

$$(M_0, g_0) \times (M_1, g_1) \times \cdots \times (M_r, g_r),$$

where (M_0, g_0) is a (possibly zero-dimensional) Euclidean space, $r \geq 0$, and each (M_i, g_i) is a non-Euclidean irreducible Riemannian manifold. Moreover, this decomposition is unique up to isometry and permutation of the factors.

The holonomy representation detects irreducibility:

Proposition 3.2 ([KN96, Theorem 6.1]). *The following are equivalent:*

(i) (M, g) *is irreducible.*

(ii) *The holonomy representation* $\mathrm{Hol}_p(M, g) \hookrightarrow \mathrm{GL}(T_pM)$ *is irreducible.*

3.1.3 Symmetric spaces

A (simply-connected) Riemannian manifold (M, g) is called a *symmetric space* if for any $p \in M$, there exists an isometry $s : M \to M$ such that $s(p) = p$ and $ds_p = -\mathrm{Id}_{T_pM}$. The isometry s is unique, because any isometry fixing p is uniquely determined by its differential at p. We call s the *geodesic symmetry at* p.

Every symmetric space (M, g) is homogeneous: the isometry group $\mathrm{Isom}(M, g)$ acts transitively on M [Bes08, 7.65]. In particular, every symmetric space is complete [Bes08, 7.19].

An *irreducible symmetric space* is a symmetric space which is irreducible as a Riemannian manifold (see §3.1.2). If the product of Riemannian manifolds is a symmetric space, then each factor is also a symmetric space [KN69, Chapter XI Theorem 6.6]. Therefore, the De Rham decomposition theorem implies that any symmetric space (M, g) decomposes uniquely into a product

$$(M_0, g_0) \times (M_1, g_1) \times \cdots \times (M_r, g_r),$$

where (M_0, g_0) is Euclidean, and (M_i, g_i) are symmetric irreducible for $i = 1, \ldots, r$. An irreducible symmetric space is of

(i) *Euclidean type* if (M, g) is Euclidean: this is equivalent to $\dim(M) = 1$.

(ii) *compact type* if M is compact.

(iii) *non-compact type* if (M, g) is not of the previous two types.

Remark 3.3. In this thesis, we are interested in irreducible symmetric spaces of non-compact type. In this case, the underlying smooth manifold M is diffeomorphic to $\mathbb{R}^n$, even when we do not assume that M is simply-connected [Hel78, Chapter VI, Theorem 1.1].

3.2 The correspondence

In this section, we describe the correspondence between symmetric spaces and simple Lie algebras over $\mathbb{R}$. We focus on the irreducible non-compact case in §3.2.2. We finish by giving algorithms for constructing the symmetric spaces (§3.2.3) and computing their isometry groups (§3.2.4).

3.2.1 From symmetric spaces to simple Lie algebras

Let (M, g) be a (not necessarily irreducible) symmetric space. Fix a point $p \in M$, and let $s : M \to M$ be the geodesic symmetry at p.

Let G denote the identity component of $\mathrm{Isom}(M, g)$. Since M is connected, it follows that G also acts transitively on M. Let K denote the isotropy subgroup of G at p. The compact subgroup K is connected: this follows immediately from [Hat02, Theorem 4.41, Page 376], since G is connected and M is simply-connected.

Let $\mathfrak{g} = \mathrm{Lie}(G)$ be the Lie algebra of G. The map $\sigma : G \to G$ given by $f \mapsto s \circ f \circ s^{-1}$ is a Lie group automorphism of G such that $\sigma^2 = \mathrm{Id}_G$. Therefore, its differential $\theta := d\sigma : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra automorphism with $\theta^2 = \mathrm{Id}_{\mathfrak{g}}$. Set

$$\mathfrak{k} = \Big\{X \in \mathfrak{g} : \theta(X) = X\Big\}, \qquad \mathfrak{p} = \Big\{X \in \mathfrak{g} : \theta(X) = -X\Big\}.$$

Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, and

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \qquad [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}, \qquad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}.$$

Moreover, the Lie algebra of K is $\mathfrak{k}$ [Arv03, Proposition 6.4].

Let $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ denote the adjoint representation of G. We have $\mathrm{Ad}(k)\mathfrak{p} \subseteq \mathfrak{p}$ for all $k \in K$ [Arv03, Proposition 6.4 (3)]. Therefore, we obtain a Lie group representation $\mathrm{Ad}|_K : K \to \mathrm{GL}(\mathfrak{p})$.

Now, let $\pi : G \to M$ denote the smooth submersion $f \mapsto f(p)$. Taking the differential at the identity of G, we obtain a surjective linear map $d\pi : \mathfrak{g} \to T_pM$. The kernel of this map is $\mathfrak{k}$, so this map restricts to a linear isomorphism $d\pi|_{\mathfrak{p}} : \mathfrak{p} \to T_pM$. This linear isomorphism is K-equivariant with respect to the adjoint representation $K \to \mathrm{GL}(\mathfrak{p})$, and the isotropy representation $K \to \mathrm{GL}(T_pM)$. Henceforth, we use this isomorphism to identify $\mathfrak{p} \cong T_pM$. The isotropy representation $K \to \mathrm{GL}(T_pM)$ (and thus $K \to \mathrm{GL}(\mathfrak{p})$) is effective, because elements of K are uniquely determined by their differential at p.

Remark 3.4. Given $X \in \mathfrak{p} \cong T_pM$, the geodesic $\gamma : \mathbb{R} \to M$ with $\gamma(0) = p$ and $\gamma'(0) = X$ is given by

$$\gamma(t) = \exp(tX) \cdot p,$$

where $\exp : \mathfrak{g} \to G$ denotes the exponential map of G [Hel78, Chapter IV Theorem 3.3 (iii)].

Proposition 3.5 ([Esc97, §§6 and 7])**.** *Assume (M, g) is irreducible. Then*

(a) *The image of the isotropy representation $K \to \mathrm{GL}(T_pM)$ is the holonomy group $\mathrm{Hol}_p(M, g)$. Thus, the adjoint representation $K \to \mathrm{GL}(\mathfrak{p})$ is irreducible.*

(b) *Under the identification $\mathfrak{p} \cong T_pM$, we can write $B|_{\mathfrak{p}\times\mathfrak{p}} = \lambda g_p$, where B is the Killing form of $\mathfrak{g}$, and $\lambda \in \mathbb{R}$. Moreover,*

  (i) *$\lambda < 0$ if and only if (M, g) is of compact type,*

  (ii) *$\lambda > 0$ if and only if (M, g) is of non-compact type.*

(c) *If (M, g) is of compact or non-compact type, then $\mathfrak{g}$ is semisimple. Moreover,*

  (i) *if (M, g) is of compact type, then $\mathfrak{g}$ is either a simple Lie algebra of compact type, or $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}$, where $\mathfrak{h}$ is a simple Lie algebra of compact type.*

  (ii) *if (M, g) is of non-compact type, then $\mathfrak{g}$ is a simple Lie algebra of non-compact type, θ is a Cartan involution, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition.*

3.2.2 Irreducible symmetric spaces of non-compact type

Proposition 3.6. *We have a bijective correspondence*

$$\left\{\begin{matrix}\text{Irreducible symmetric spaces}\\ \text{of non-compact type}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Simple Lie algebras}\\ \text{of non-compact type}\end{matrix}\right\}$$

given by $(M, g) \mapsto \mathrm{Lie}\,\mathrm{Isom}(M, g)$.

Remark 3.7. The collection on the left is considered up to isometry and scaling. The collection on the right is considered up to isomorphisms. The Lie algebras here are real.

Proof of Proposition 3.6. First, let us show that the mapping $(M, g) \mapsto \mathrm{Lie}\,\mathrm{Isom}(M, g)$ is well-defined. By Proposition 3.5 Part (c), we know that $\mathrm{Lie}\,\mathrm{Isom}(M, g)$ is indeed a simple Lie algebra of non-compact type. Moreover, if two Riemannian manifolds are homothetic, then the Lie algebras of their isometry groups are isomorphic.

Injectivity follows immediately from [Hel78, Chapter VI Corollary 1.3] and Proposition 3.5 Part (b).

For surjectivity, let $\mathfrak{g}$ be a simple Lie algebra of non-compact type. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ a Cartan decomposition. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ whose center $Z(G)$ is finite (such a G exists by [Kna96, Proposition 6.30]). Let K be the unique connected Lie subgroup of G whose Lie algebra is $\mathfrak{k}$. Then [Kna96, Theorem 6.31(f)] implies that K is compact. Set $M := G/K$. Since K is compact, it follows that there exists a G-invariant metric g on M. [Hel78, Chapter VI Theorem 1.1] and [Hel78, Chapter IV Proposition 3.4] imply that M is simply-connected, and (M, g) is a symmetric space.

Let us show that $\mathrm{Lie}\,\mathrm{Isom}(M, g)$ is isomorphic to $\mathfrak{g}$. Let $\tau : G \to \mathrm{Isom}(M, g)$ be the Lie group isomorphism induced from the action of G on M. The kernel of τ is equal to the center of G. It is easy to see that $Z(G) \subseteq \ker\tau$. For the other inclusion, observe that $\ker\tau$ is a normal Lie subgroup of G, so $\mathrm{Lie}\ker\tau$ is an ideal of $\mathfrak{g}$. Since $\mathfrak{g}$ is simple and τ is not trivial, it follows that $\mathrm{Lie}\ker\tau$ is trivial. Thus, $\ker\tau$ is a discrete normal subgroup of G. Therefore, [Lee13, Problem 21-18] tells us that $\ker\tau \subseteq Z(G)$. By [Hel78, Chapter V Theorem 4.1 Part (i)] and the fact that $Z(G)$ is discrete, it follows that the image of τ is the identity component of $\mathrm{Isom}(M, g)$. Therefore, $\mathrm{Lie}\,\mathrm{Isom}(M, g)$ is isomorphic to $\mathfrak{g}$ as we wished.

It remains to show that (M, g) is irreducible. For the sake of contradiction, suppose (M, g) is not irreducible. We know (M, g) is isometric to a product $(M_0, g_0) \times (M_1, g_1) \times \cdots \times (M_r, g_r)$, where (M_0, g_0) is a (possibly zero-dimensional) Euclidean space, and (M_i, g_i) are irreducible symmetric spaces of compact type or non-compact type for $i = 1, \ldots, r$. [Wol11, Theorem 8.3.9] implies that $\mathfrak{g}$ is isomorphic to the direct sum of the Lie algebras of the isometry groups of (M_i, g_i) for $i = 1, \ldots, r$. Since we assumed (M, g) is not irreducible, it follows that $\mathfrak{g}$ is not simple. This is a contradiction. □

Proposition 3.8 ([Hel78, Chapter VI Exercise 7])**.** *Let (M, g) be an irreducible symmetric space of non-compact type, and let $\mathfrak{g} := \operatorname{Lie}\operatorname{Isom}(M, g)$. Then the adjoint representation*

$$\mathrm{Ad} : \operatorname{Isom}(M, g) \to \operatorname{Aut}(\mathfrak{g})$$

is a Lie group isomorphism.

Proof. We show injectivity. First, since (M, g) is of non-compact type, we know that (M, g) is a Cartan-hadamard manifold: it is simply-connected and has non-positive curvature [Esc97, §6]. Thus, between any two points in M, there is a unique geodesic passing through both points. In particular, any isometry fixing two points fixes the geodesic passing through both points.

Now, suppose f belongs to $\ker(\mathrm{Ad})$. Then f commutes with every element in $G = \operatorname{Isom}(M, g)^0$. Fix any $p \in M$, and let us show that $f(p) = p$. For the sake of contradiction, suppose $f(p) \neq p$. Since f commutes with every element of K, the isotropy subgroup at p, it follows that every element of K fixes $f(p)$. Therefore, every element of K fixes the geodesic $\gamma : \mathbb{R} \to M$ satisfying $\gamma(0) = p$ and passing through $f(p)$. Let $\gamma'(0) = X \in T_pM$. Then $a := \mathbb{R}X$ is an invariant subspace of T_pM with respect to the isotropy action by K. Since (M, g) is irreducible, this contradicts Proposition 3.5 Part (a). Therefore, $f(p) = p$. □

Proposition 3.9 ([Och69])**.** *Let H be a Lie group acting smoothly on the irreducible symmetric space of non-compact type (M, g). If the image of $H \to \operatorname{Diff}(M)$ contains the identity component $\operatorname{Isom}(M, g)^0$, then H acts by isometries.*

3.2.3 Algorithm for constructing (M, g)

The proof of Proposition 3.6 gives us an algorithm for constructing the irreducible symmetric space of non-compact type corresponding to a simple Lie algebra of non-compact type:

Algorithm 3.10. Let $\mathfrak{g}$ be a simple Lie algebra of non-compact type.

(i) Choose a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

(ii) Choose a Lie group G with finite center and Lie algebra $\mathfrak{g}$, and let K denote the connected Lie subgroup with Lie algebra $\mathfrak{k}$.

(iii) Choose a Riemannian manifold (M, g), and a smooth, transitive and isometric action of G on (M, g) with isotropy K at a point $p \in M$.

Then (M, g) is the irreducible symmetric space of non-compact type corresponding to $\mathfrak{g}$.

For the Lie algebras $\mathfrak{g}$ of interest, we have already completed Step (i) in Table 2.9, which gives the Cartan decompositions $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. The following table completes Step (ii):

$\mathfrak{g}$	G	K
$\mathfrak{sl}(n, \mathbb{R})$	$\mathrm{SL}(n, \mathbb{R})$	$\mathrm{SO}(n)$
$\mathfrak{sl}(n, \mathbb{C})$	$\mathrm{SL}(n, \mathbb{C})$	$\mathrm{SU}(n)$
$\mathfrak{sl}(n, \mathbb{H})$	$\mathrm{SL}(n, \mathbb{H})$	$\mathrm{Sp}(n)$
$\mathfrak{so}(p, q)$	$\mathrm{O}(p, q)^0$	$\mathrm{SO}(p) \times \mathrm{SO}(q)$
$\mathfrak{su}(p, q)$	$\mathrm{SU}(p, q)$	$S(\mathrm{U}(p) \times \mathrm{U}(q))$
$\mathfrak{sp}(p, q)$	$\mathrm{Sp}(p, q)$	$\mathrm{Sp}(p) \times \mathrm{Sp}(q)$
$\mathfrak{sp}(2n, \mathbb{R})$	$\mathrm{Sp}(2n, \mathbb{R})$	$\Phi(\mathrm{U}(n))$
$\mathfrak{sp}(2n, \mathbb{C})$	$\mathrm{Sp}(2n, \mathbb{C})$	$\Psi(\mathrm{Sp}(n))$
$\mathfrak{so}(n, \mathbb{C})$	$\mathrm{SO}(n, \mathbb{C})$	$\mathrm{SO}(n)$
$\mathfrak{so}(n, \mathbb{H})$	$\mathrm{SO}(n, \mathbb{H})$	$\mathrm{U}(n)$

Table 3.1: A connected Lie group G with finite center whose Lie algebra is $\mathfrak{g}$, and the connected Lie subgroup K whose Lie algebra is $\mathfrak{k}$ given in Table 2.9. Here, Φ and Ψ denote the embeddings defined in §2.1.4.

We complete Step (iii) of the algorithm in the subsequent sections. In each case, we realise (M, g) as either a space of positive-definite matrices, or as an open subset of a matrix vector space.

3.2.4 Algorithm for computing $\mathrm{Isom}(M, g)$

Let $\mathfrak{g}$ be a simple Lie algebra of non-compact type listed in Table 3.1, and let G and K denote the corresponding connected Lie groups in Table 3.1. In the subsequent sections, we complete Step (iii) of Algorithm 3.10. That is, we construct the symmetric space (M, g) corresponding to $\mathfrak{g}$, and equip it with an action of G on M.

Recall that the adjoint representation $\mathrm{Ad} : \mathrm{Isom}(M, g) \to \mathrm{Aut}(\mathfrak{g})$ of $\mathrm{Isom}(M, g)$ is an isomorphism. Therefore, it suffices to "guess" a collection of isometries S, then show that

$$\mathrm{Ad}(S) = \mathrm{Isom}(M, g).$$

Recall that in §2.4, we describe the automorphism group of $\mathfrak{g}$ as

$$\mathrm{Aut}(\mathfrak{g}) = \mathrm{Ad}'(G') \cdot \Gamma', \tag{3.1}$$

where

(i) G' is the matrix Lie group with identity component G given in Table 2.10,

(ii) $\mathrm{Ad}' : G' \to \mathrm{Aut}(\mathfrak{g})$ denotes the adjoint representation of G', and

(iii) Γ' is the finite subgroup of $\mathrm{Aut}(\mathfrak{g})$ given in Table 2.10.

In the subsequent sections, our "guesses" of $\mathrm{Isom}(M, g)$ are of the form

$$S = \tau(G') \cdot \Gamma,$$

where

(i) G' is the same Lie group given in Table 2.10 as before,

(ii) $\tau : G' \to \mathrm{Isom}(M,g)$ is a smooth and isometric action of G' on (M,g) which extends the action of G, and

(iii) Γ is a finite subgroup of $\mathrm{Isom}(M,g)$.

We find that

$$\mathrm{Ad}(S) = \mathrm{Ad}(\tau(G')\cdot\Gamma) = \mathrm{Ad}(\tau(G'))\cdot\mathrm{Ad}(\Gamma) = \mathrm{Ad}'(G')\cdot\mathrm{Ad}(\Gamma),$$

where the last equality holds because the following diagram commutes:

$$\begin{tikzcd} G' \arrow[d, "\tau"'] \arrow[dr, "\mathrm{Ad}'"] & \\ \mathrm{Isom}(M,g) \arrow[r, "\mathrm{Ad}"'] & \mathrm{Aut}(\mathfrak{g}) \end{tikzcd}$$

Thus, in light of (3.1), it suffices to show that $\mathrm{Ad}(\Gamma) = \Gamma'$.

Remark 3.11. Thanks to Proposition 3.9, we have a shortcut to show that a certain diffeomorphism $f : M \to M$ is an isometry: it suffices to show that f normalises $\tau(G')$.

3.3 The cases $\mathfrak{sl}(n,\mathbb{R})$, $\mathfrak{sl}(n,\mathbb{C})$ and $\mathfrak{sl}(n,\mathbb{H})$

In this section, we construct the symmetric spaces corresponding to $\mathfrak{sl}(n,\mathbb{K})$ for $\mathbb{K} = \mathbb{R},\mathbb{C},\mathbb{H}$, then compute their isometry groups. Throughout this section, assume $n \geq 2$ and $\mathbb{K} = \mathbb{R},\mathbb{C},\mathbb{H}$. A reference for the case $\mathbb{K} = \mathbb{R}$ is [BH99, Chapter 10].

3.3.1 The space of positive-definite matrices

Let $\mathcal{P}(n,\mathbb{K})$ denote the set of positive-definite matrices:

$$\mathcal{P}(n,\mathbb{K}) := \Big\{P \in \mathrm{Herm}(n,\mathbb{K}) : P > 0\Big\}.$$

The set $\mathcal{P}(n,\mathbb{K})$ is an open subset of the real vector space $\mathrm{Herm}(n,\mathbb{K})$.

Proposition 3.12. *The Lie group* $\mathrm{GL}(n,\mathbb{K})$ *acts smoothly on* $\mathcal{P}(n,\mathbb{K})$ *via* $A\cdot P = APA^*$. *Moreover,*

(i) *This action is transitive.*

(ii) *The isotropy subgroup at* I_n *is* $\mathrm{GL}(n,\mathbb{K})\cap\mathrm{Sp}(n)$.

Proof. It is easy to check that APA^* is again positive-definite Hermitian whenever $A \in \mathrm{GL}(n,\mathbb{K})$ and $P \in \mathcal{P}(n,\mathbb{K})$. Clearly $A\cdot P = APA^*$ defines a smooth group action. Part (ii) is obvious.

For transitivity, fix $P \in \mathcal{P}(n,\mathbb{K})$. By the spectral theorem (see [FP03] for the quaternionic case), we can write $P = SDS^*$, where S is unitary and $D = \mathrm{diag}(\lambda_1,\dots,\lambda_n)$, where $\lambda_1,\dots,\lambda_n$ are the eigenvalues of P, which are positive because P is positive-definite. Set $A := S\sqrt{D}S^*$, where $\sqrt{D} := \mathrm{diag}(\sqrt{\lambda_1},\dots,\sqrt{\lambda_n})$. Then $A\cdot\mathrm{I}_n = S\sqrt{D}S^*S\sqrt{D}S^* = P$, where the last equality follows because S is unitary. ☐

Proposition 3.13. *On* $T_{\mathrm{I}_n}\mathcal{P}(n,\mathbb{K}) \cong \mathrm{Herm}(n,\mathbb{K})$*, define* $\langle V,W\rangle := \mathrm{Re}\,\mathrm{tr}(VW)$*. Then*

(i) *$\langle\cdot,\cdot\rangle$ is a real inner product which is* $\mathrm{GL}(n,\mathbb{K})\cap\mathrm{Sp}(n)$*-invariant with respect to the isotropy representation.*

(ii) *Let g be the unique* $\mathrm{GL}(n,\mathbb{K})$*-invariant Riemannian metric on $\mathcal{P}(n,\mathbb{K})$ with $g_{\mathrm{I}_n}=\langle\cdot,\cdot\rangle$. Then*

$$g_P(V,W)=\operatorname{Re}\operatorname{tr}(P^{-1}VP^{-1}W)$$

for all $P\in\mathcal{P}(n,\mathbb{K})$ and $V,W\in T_P\mathcal{P}(n,\mathbb{K})\cong\mathrm{Herm}(n,\mathbb{K})$.

Proof. For each $A\in\mathrm{GL}(n,\mathbb{K})$, let $\tau(A):\mathcal{P}(n,\mathbb{K})\to\mathcal{P}(n,\mathbb{K})$ denote the map $P\mapsto APA^*$. The differential at P is given by $d\tau(A)_P:V\mapsto AVA^*$. The map $\operatorname{Re}\operatorname{tr}$ is cyclic: if X and Y are matrices, then $\operatorname{Re}\operatorname{tr}(XY)=\operatorname{Re}\operatorname{tr}(YX)$. From this fact, it is easy to see that $\langle\cdot,\cdot\rangle$ is a real inner product. Now, fix $A\in\mathrm{GL}(n,\mathbb{K})\cap\mathrm{Sp}(n)$, and $V,W\in T_{\mathrm{I}_n}\mathcal{P}(n,\mathbb{K})$. Then

$$\langle d\tau(A)V,d\tau(A)W\rangle=\operatorname{Re}\operatorname{tr}(AVA^*AWA^*)=\operatorname{Re}\operatorname{tr}(V,W)=\langle V,W\rangle,$$

where the second equality holds because A is unitary.

To show (ii), fix any $P\in\mathcal{P}(n,\mathbb{K})$, and choose $A\in\mathrm{GL}(n,\mathbb{K})$ so that $AA^*=P$. Fix $V,W\in T_P\mathcal{P}(n,\mathbb{K})$. Then

$$\begin{aligned}g_P(V,W)&=g_{\mathrm{I}_n}(d\tau(A^{-1})V,d\tau(A^{-1})W)\\&=\operatorname{Re}\operatorname{tr}(A^{-1}V(A^{-1})^*A^{-1}W(A^{-1})^*)=\operatorname{Re}\operatorname{tr}(P^{-1}VP^{-1}W),\end{aligned}$$

as desired. □

Henceforth, we equip $\mathcal{P}(n,\mathbb{K})$ with the Riemannian metric g.

Remark 3.14. The Riemannian manifold $\mathcal{P}(n,\mathbb{K})$ is a symmetric space: the geodesic symmetry at I_n is given by $s:P\mapsto P^{-1}$.

3.3.2 The symmetric space

In this subsection, we construct $\mathcal{P}(n,\mathbb{K})_1$, the irreducible symmetric space of non-compact type corresponding to $\mathfrak{sl}(n,\mathbb{K})$. The underlying smooth manifold is the embedded submanifold of $\mathcal{P}(n,\mathbb{K})$ of codimension one given by

$$\mathcal{P}(n,\mathbb{K})_1:=\mathcal{P}(n,\mathbb{K})\cap\mathrm{SL}(n,\mathbb{K}).$$

We equip $\mathcal{P}(n,\mathbb{K})_1$ with the Riemannian metric induced by the Riemannian metric on $\mathcal{P}(n,\mathbb{K})$:

$$g_P(V,W)=\operatorname{Re}\operatorname{tr}(P^{-1}VP^{-1}W)$$

for $P\in\mathcal{P}(n,\mathbb{K})_1$ and $V,W\in T_P\mathcal{P}(n,\mathbb{K})_1\subseteq\mathrm{Herm}(n,\mathbb{K})$.

Remark 3.15. The map $\mathbb{R}\times\mathcal{P}(n,\mathbb{K})_1\to\mathcal{P}(n,\mathbb{K})$ given by $(s,P)\mapsto e^{s/\sqrt{n}}P$ is a Riemannian isometry. In particular, $\mathcal{P}(n,\mathbb{K})_1$ is totally geodesic in $\mathcal{P}(n,\mathbb{K})$.

Proposition 3.16. *Consider the action of* $\mathrm{SL}(n,\mathbb{K})$ *on $\mathcal{P}(n,\mathbb{K})$ given by $A\cdot P=APA^*$.*

(i) *The orbit of* I_n *for this action is $\mathcal{P}(n,\mathbb{K})_1$.*

(ii) *The isotropy subgroup at* I_n *is* $K=\mathrm{SL}(n,\mathbb{K})\cap\{\text{Unitary matrices}\}$.

The proof is the same as the proof of Proposition 3.12. Thus, by Algorithm 3.10, we have shown the following:

Proposition 3.17. *The Riemannian manifold $\mathcal{P}(n,\mathbb{K})_1$ is the irreducible symmetric space corresponding to the Lie algebra $\mathfrak{sl}(n,\mathbb{K})$.*

Remark 3.18. When $\mathbb{K}=\mathbb{R}$, the orbit of I_n with respect to the Lie group $\mathrm{SL}_{\pm}(n,\mathbb{R})$ is also $\mathcal{P}(n,\mathbb{R})_1$.

3.3.3 The isometry group

Proposition 3.19. *The isometry group of* $\mathcal{P}(n,\mathbb{K})_1$ *is given by*

$$\operatorname{Isom}(\mathcal{P}(n,\mathbb{K})_1) = \tau(G')\cdot\Gamma,$$

where G' *and* Γ *are given by the following table, and* $\tau : G' \to \operatorname{Isom}(\mathcal{P}(n,\mathbb{K})_1)$ *is the action of* G' *on* $\mathcal{P}(n,\mathbb{K})_1$ *given by* $A\cdot P = APA^*$*.*

$\mathbb{K}$	G'	Γ
$\mathbb{R}$	$\mathrm{SL}_\pm(n,\mathbb{R})$	$P\mapsto P^{-1}$ when $n\geq 3$
$\mathbb{C}$	$\mathrm{SL}(n,\mathbb{C})$	$P\mapsto P^{-1}$ when $n\geq 3$, $P\mapsto \overline{P}$
$\mathbb{H}$	$\mathrm{SL}(n,\mathbb{H})$	$P\mapsto P^{-1}$ when $n\geq 2$

Proof. By the discussion in §3.2.4, it suffices to show that Γ normalises $\tau(G')$ and $\Gamma' = \operatorname{Ad}(\Gamma)$, where Γ' is the finite subgroup of $\operatorname{Aut}(\mathfrak{sl}(n,\mathbb{K}))$ given in Table 2.10. Let $s : \mathcal{P}(n,\mathbb{K})_1 \to \mathcal{P}(n,\mathbb{K})_1$ denote the geodesic symmetry $P\mapsto P^{-1}$, and let $c : \mathcal{P}(n,\mathbb{C})_1 \to \mathcal{P}(n,\mathbb{C})_1$ denote the map $P\mapsto\overline{P}$. Given $A\in G'$, we find

$$s\circ\tau(A)\circ s^{-1} = \tau((A^{-1})^*), \qquad c\circ\tau(A)\circ c^{-1} = \tau(\overline{A}).$$

Therefore, $\operatorname{Ad}(s) : \mathfrak{sl}(n,\mathbb{K}) \to \mathfrak{sl}(n,\mathbb{K})$ and $\operatorname{Ad}(c) : \mathfrak{sl}(n,\mathbb{C}) \to \mathfrak{sl}(n,\mathbb{C})$ are given by

$$\operatorname{Ad}(s) : X\mapsto -X^*, \qquad \operatorname{Ad}(c) : X\mapsto \overline{X},$$

so $\Gamma' = \operatorname{Ad}(\Gamma)$, as desired. □

3.4 The cases $\mathfrak{so}(p,q)$, $\mathfrak{su}(p,q)$, and $\mathfrak{sp}(p,q)$

In this section, we construct the symmetric spaces corresponding to $\mathfrak{so}(p,q)$, $\mathfrak{su}(p,q)$, and $\mathfrak{sp}(p,q)$, then compute their isometry groups. Assume $p,q\geq 1$, $\mathbb{K}=\mathbb{R},\mathbb{C},\mathbb{H}$, and write $n=p+q$. A reference for the case $q=1$ is [BH99, Chapter 10].

3.4.1 Grassmannians

Recall that the *Grassmannian* $\mathrm{Gr}_q(\mathbb{K}^n)$ is defined as

$$\mathrm{Gr}_q(\mathbb{K}^n) := \Big\{q\text{-dimensional } \mathbb{K}\text{-subspaces of } \mathbb{K}^n\Big\}.$$

In the case $\mathbb{K}=\mathbb{H}$, we declare scalars $\mathbb{H}$ to act on $\mathbb{H}^n$ from the right. The Lie group $\mathrm{GL}(n,\mathbb{K})$ acts transitively on $\mathrm{Gr}_q(\mathbb{K}^n)$ via $L\cdot V := L(V)$. We equip $\mathrm{Gr}_q(\mathbb{K}^n)$ with the unique smooth structure such that the action of $\mathrm{GL}(n,\mathbb{K})$ is smooth [Lee13, Example 21.21]. Now, let

$$\mathcal{U} := \Big\{V\in\mathrm{Gr}_q(\mathbb{K}^n) : V\cap(\mathbb{K}^p\oplus 0) = 0\Big\}.$$

Then $\mathcal{U}$ is an open subset of $\mathrm{Gr}_q(\mathbb{K}^n)$, and the map $F : M_{p\times q}(\mathbb{K}) \to \mathcal{U}$ given by

$$F(Z) := \left\{\begin{pmatrix} Zv \\ v\end{pmatrix} : v\in\mathbb{K}^q\right\} \tag{3.2}$$

is a diffeomorphism [Lee13, Example 1.36, Problem 21-12]. The inverse of F is given by the following. Given $V\in\mathcal{U}$, there exist unique $v_1,\dots,v_q\in\mathbb{K}^p$ such that $(v_1,e_1),\dots,(v_q,e_q)$ belong to V, where $e_1,\dots,e_q$ are the standard basis vectors of $\mathbb{K}^q$. Then $F^{-1}(V)$ is the p by q matrix whose columns are $v_1,\dots,v_q$.

3.4.2 The symmetric space

In this subsection, we define the *hyperbolic Grassmannian* $\mathcal{H}_{p,q}(\mathbb{K})$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, which are the irreducible symmetric spaces of non-compact type corresponding to $\mathfrak{so}(p, q)$, $\mathfrak{su}(p, q)$, and $\mathfrak{sp}(p, q)$, respectively. The underlying smooth manifold of $\mathcal{H}_{p,q}(\mathbb{K})$ is the open subset of $\mathrm{M}_{p\times q}(\mathbb{K})$ defined by

$$\mathcal{H}_{p,q}(\mathbb{K}) := \Big\{ Z \in \mathrm{M}_{p\times q}(\mathbb{K}) : Z^*Z - \mathrm{I}_q < 0 \Big\},$$

where $Z^*Z - \mathrm{I}_q < 0$ means the square matrix $Z^*Z - \mathrm{I}_q$ is negative-definite. Note that the condition $Z^*Z - \mathrm{I}_q$ is equivalent to every eigenvalue λ of Z^*Z satisfying $\lambda < 1$. Recall that the *operator norm* on $\mathrm{M}_{p\times q}(\mathbb{K})$ is defined by

$$\|Z\|_{\mathrm{op}} := \max_{v\in\mathbb{K}^n : |v|=1} |Zv|,$$

where $|\cdot|$ here denotes the standard norms on $\mathbb{K}^p$ and $\mathbb{K}^q$. This norm satisfies $\|Z\|_{\mathrm{op}} = \sqrt{\lambda_{\max}}$, where $\lambda_{\max}$ is the largest eigenvalue of Z^*Z [Mey00, §5.2, Page 281]. Therefore, the condition $Z^*Z - \mathrm{I}_q$ is equivalent to $\|Z\|_{\mathrm{op}} < 1$, and

$$\mathcal{H}_{p,q}(\mathbb{K}) = \Big\{ Z \in \mathrm{M}_{p\times q}(\mathbb{K}) : \|Z\|_{\mathrm{op}} < 1 \Big\}.$$

In words, the smooth manifold $\mathcal{H}_{p,q}(\mathbb{K})$ is precisely the open unit ball in the normed space $(\mathrm{M}_{p\times q}(\mathbb{K}), \|\cdot\|_{\mathrm{op}})$.

Remark 3.20. For $Z \in \mathrm{M}_{p\times q}(\mathbb{K})$, the sets of non-zero eigenvalues of Z^*Z and ZZ^* agree. Therefore, $Z^*Z - \mathrm{I}_q < 0$ if and only if $ZZ^* - \mathrm{I}_p < 0$.

Let h denote the standard indefinite Hermitian form with signature (p, q) on $\mathbb{K}^n$. Recall that

$$h(v, w) = v^*\, \mathrm{I}_{p,q}\, w, \qquad \text{where } \mathrm{I}_{p,q} = \begin{pmatrix} \mathrm{I}_p & 0 \\ 0 & -\mathrm{I}_q \end{pmatrix}.$$

Proposition 3.21. *Let $F : \mathrm{M}_{p\times q}(\mathbb{K}) \to \mathrm{Gr}_q(\mathbb{K}^n)$ denote the embedding (3.2). The image of $\mathcal{H}_{p,q}(\mathbb{K})$ under F is*

$$F(\mathcal{H}_{p,q}(\mathbb{K})) = \Big\{ V \in \mathrm{Gr}_q(\mathbb{K}^n) : h|_{V\times V} < 0 \Big\}.$$

Here, $h|_{V\times V} < 0$ means that the restriction of h to V is negative-definite.

Proof of Proposition 3.21. First, fix $V \in \mathrm{Gr}_q(\mathbb{K}^n)$ with $h|_{V\times V} < 0$, and let us show that V belongs to $\mathcal{U}$. Suppose $w \in \mathbb{K}^p$ and $(w, 0) \in V$. Then $0 \leq w^*w = h((w, 0), (w, 0)) \leq 0$, so $w = 0$. Thus, $V \cap (\mathbb{K}^p \oplus 0) = 0$, as desired.

Next, for $Z \in \mathrm{M}_{p\times q}(\mathbb{K})$, observe that the following statements are equivalent:

(i) $Z^*Z - \mathrm{I}_q < 0$.

(ii) $h((Zv, v), (Zv, v)) = v^*(Z^*Z - \mathrm{I}_q)v < 0$ for all non-zero $v \in \mathbb{K}^q$.

(iii) h is negative-definite on $F(Z)$.

This completes the proof. □

Now, let G' and K' be the Lie groups given by the following table:

$\mathbb{K}$	G'	K'
$\mathbb{R}$	$\mathrm{O}(p,q)$	$\mathrm{O}(p)\times\mathrm{O}(q)$
$\mathbb{C}$	$\mathrm{SU}(p,q)$	$S(\mathrm{U}(p)\times\mathrm{U}(q))$
$\mathbb{H}$	$\mathrm{Sp}(p,q)$	$\mathrm{Sp}(p)\times\mathrm{Sp}(q)$

Proposition 3.22. *Consider the action of G' on $\mathrm{Gr}_q(\mathbb{K}^n)$ given by $L\cdot V := L(V)$.*

(i) *The orbit of $0\oplus\mathbb{K}^q$ is $F(\mathcal{H}_{p,q}(\mathbb{K}))$.*

(ii) *The isotropy subgroup at $0\oplus\mathbb{K}^q$ is K'.*

Proof. To show (i), suppose $L\in G'$. Fix a non-zero element $Lv\in L(V)$, so that $v\in 0\oplus\mathbb{K}^q$. Then $h(Lv,Lv)=h(v,v)<0$, so $L(V)$ belongs to $F(\mathcal{H}_{p,q}(\mathbb{K}))$. Conversely, suppose $V\in\mathrm{Gr}_q(\mathbb{K}^n)$ and h is negative-definite on V. Let $W:=V^\top$ denote the orthogonal complement of V with respect to h. Sylvester's law of inertia (see [Ros02, Appendix to §3.1]) tells us that $\mathbb{K}^n=W\oplus V$ and h is positive-definite on W. Let $v_1,\dots,v_p$ be an orthonormal basis for W with respect to h, and let $v_{p+1},\dots,v_n$ be an orthonormal basis for V with respect to $-h$. Let L be the n by n matrix whose columns are $v_1,\dots,v_n$. Then L belongs to G' (when $\mathbb{K}=\mathbb{C}$, we may need to multiply v_1 by some $\lambda\in\mathbb{S}^1$ to ensure that $\det(L)=1$). Moreover, $L(0\oplus\mathbb{K}^q)=\mathrm{span}_{\mathbb{K}}\{v_{p+1},\dots,v_n\}=V$.

To show (ii), let $L\in G'$, and suppose that $L(0\oplus\mathbb{K}^q)=0\oplus\mathbb{K}^q$. Since L preserves orthogonal complements with respect to h, it follows that $L(\mathbb{K}^p\oplus 0)=\mathbb{K}^p\oplus 0$. Write L in block matrix form, $L=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)$. By definition of G', we can write

$$L^{-1}=\begin{pmatrix}A^* & -C^*\\ -B^* & D^*\end{pmatrix},$$

so it suffices to show that $B=0$ and $C=0$. Given $v\in\mathbb{K}^q$, we find

$$\begin{pmatrix}0\\ *\end{pmatrix}=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}0\\ v\end{pmatrix}=\begin{pmatrix}Cv\\ Dv\end{pmatrix},$$

so $C=0$. A similar argument using $L(\mathbb{K}^p\oplus 0)=\mathbb{K}^p\oplus 0$ shows that $B=0$. □

We equip $\mathcal{H}_{p,q}(\mathbb{K})$ with the action of G' obtained by the diffeomorphism $F:\mathcal{H}_{p,q}(\mathbb{K})\cong F(\mathcal{H}_{p,q}(\mathbb{K}))$.

Proposition 3.23. *The action of G' on $\mathcal{H}_{p,q}(\mathbb{K})$ is given by*

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix}\cdot Z=(AZ+B)(CZ+D)^{-1}.$$

Moreover, this action is transitive, and the isotropy subgroup at $0\in\mathcal{H}_{p,q}(\mathbb{K})$ is K'.

Proof. The second sentence follows immediately from 3.22, since $F(0)=0\oplus\mathbb{K}^q$.

To show the first sentence, fix any $Z\in\mathcal{H}_{p,q}(\mathbb{K})$, and $L=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)$ in G'. Then

$$L\cdot F(Z)=\left\{\begin{pmatrix}(AZ+B)v\\ (CZ+D)v\end{pmatrix} : v\in\mathbb{K}^q\right\}.$$

Since $L\cdot F(Z)$ belongs to $\mathcal{U}$ by Proposition 3.22, there must exist $v_1,\dots,v_q\in\mathbb{K}^q$ such that $(CZ+D)v_i=e_i$. Thus, $CZ+D$ is surjective, so $CZ+D$ is invertible. Therefore,

$$L\cdot F(Z)=\left\{\begin{pmatrix}(AZ+B)(CZ+D)^{-1}v\\ v\end{pmatrix} : v\in\mathbb{K}^q\right\}=F((AZ+B)(CZ+D)^{-1}),$$

as desired. □

Proposition 3.24. *On $T_0\mathcal{H}_{p,q}(\mathbb{K}) \cong \mathrm{M}_{p\times q}(\mathbb{K})$, define $\langle V, W\rangle := \operatorname{Re}\operatorname{tr}(VW^*)$. Then*

(i) *$\langle\cdot,\cdot\rangle$ is a real inner product which is K'-invariant with respect to the isotropy representation.*

(ii) *Let g denote the unique G'-invariant metric on $\mathcal{H}_{p,q}(\mathbb{K})$ with $g_0 = \langle\cdot,\cdot\rangle$. Then*

$$g_Z(V,W) = \operatorname{Re}\operatorname{tr}\left((\mathrm{I}_p - ZZ^*)^{-1} V (\mathrm{I}_q - Z^*Z)^{-1} W^*\right)$$

for all $Z \in \mathbb{H}_{p,q}(\mathbb{K})$ and $V, W \in T_Z\mathbb{H}_{p,q}(\mathbb{K}) \cong \mathrm{M}_{p\times q}(\mathbb{K})$.

Proof. For each $L = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)$ in G', let $\tau(L) : \mathcal{H}_{p,q}(\mathbb{K}) \to \mathcal{H}_{p,q}(\mathbb{K})$ denote the map $L \mapsto L\cdot Z$. A straightforward computation shows that the differential of $\tau(L)$ at Z is given by

$$d\tau(L)_Z : V \mapsto (A - (L\cdot Z)C)V(CZ + D)^{-1}.$$

Let us show (i). It is easy to see that $\langle\cdot,\cdot\rangle$ is a real inner product. To see that it is K'-invariant, suppose L belongs to K'. Then $B = 0$ and $C = 0$, and A, D are unitary. Therefore, given $V, W \in T_0\mathcal{H}_{p,q}(\mathbb{K})$, we find

$$\langle d\tau(L)_0 V, d\tau(L)_0 W\rangle = \operatorname{Re}\operatorname{tr}(AVD^*DW^*A^*) = \operatorname{Re}\operatorname{tr}(VW^*) = \langle V, W\rangle.$$

To show (ii), first fix $Z \in \mathcal{H}_{p,q}(\mathbb{K})$. Let L be the block matrix

$$L := \begin{pmatrix} P & ZQ \\ Z^*P & Q \end{pmatrix},$$

where $P := \left(\sqrt{\mathrm{I}_p - ZZ^*}\right)^{-1}$ and $Q := \left(\sqrt{\mathrm{I}_q - Z^*Z}\right)^{-1}$: recall that if A is a positive-semidefinite Hermitian matrix, then there exists a unique positive-semidefinite Hermitian matrix $\sqrt{A}$ such that $\sqrt{A}\sqrt{A} = A$. A long but straightforward computation shows that L belongs to G'. Clearly $L\cdot 0 = Z$. Since L belongs to G', we know that $L^{-1} = \left(\begin{smallmatrix} P & -PZ \\ -QZ^* & Q\end{smallmatrix}\right)$, so $d\tau(L^{-1})_Z V = PVQ$. Therefore, given $V, W \in T_Z\mathcal{H}_{p,q}(\mathbb{K})$, we find

$$g_Z(V,W) = g_{\mathrm{I}_n}(d\tau(L^{-1})_Z V, d\tau(L^{-1})_Z W) = \operatorname{Re}\operatorname{tr}(PVQQW^*P) = \operatorname{Re}\operatorname{tr}(P^2VQ^2W^*),$$

as desired. □

Henceforth, we equip $\mathcal{H}_{p,q}(\mathbb{K})$ with the Riemannian metric g given by Proposition 3.24. The Riemannian manifold $\mathcal{H}_{p,q}(\mathbb{K})$ is called a *hyperbolic Grassmannian.*

Remark 3.25. When $q = 1$, the Riemannian manifolds $\mathcal{H}_{n,1}(\mathbb{R})$, $\mathcal{H}_{n,1}(\mathbb{C})$ and $\mathcal{H}_{n,1}(\mathbb{H})$ are called the real, complex and quaternionic hyperbolic spaces, respectively [BH99, Chapter 10] (also see [Lee24, Problem 8-5]). In particular, $\mathcal{H}_{n,1}(\mathbb{R})$ is the Beltrami-Klein model of n-dimensional real hyperbolic space [Lee18, Theorem 3.7(b)].

The following proposition follows immediately from Proposition 3.23 and Algorithm 3.10:

Proposition 3.26. *The Riemannian manifolds $\mathcal{H}_{p,q}(\mathbb{R})$, $\mathcal{H}_{p,q}(\mathbb{C})$ and $\mathcal{H}_{p,q}(\mathbb{H})$ are the irreducible symmetric spaces of non-compact type corresponding to the Lie algebras $\mathfrak{so}(p,q)$, $\mathfrak{su}(p,q)$, and $\mathfrak{sp}(p,q)$, respectively.*

3.4.3 The isometry group

Proposition 3.27. *Ignoring the cases* $\mathbb{K}=\mathbb{R}$ *with* $(p,q)=(1,1),(2,2),(4,4)$, *the isometry group of* $\mathcal{H}_{p,q}(\mathbb{K})$ *is given by*

$$\mathrm{Isom}(\mathcal{H}_{p,q}(\mathbb{K}))=\tau(G')\cdot\Gamma,$$

where G' *and* Γ *are given by the following table, and* $\tau:G'\to\mathrm{Isom}(\mathcal{H}_{p,q}(\mathbb{K}))$ *is the action of* G' *on* $\mathcal{H}_{p,q}(\mathbb{K})$ *given in Proposition 3.23.*

$\mathbb{K}$	G'	Γ
$\mathbb{R}$	$\mathrm{O}(p,q)$	$Z\mapsto Z^\top$ when $p=q$
$\mathbb{C}$	$\mathrm{SU}(p,q)$	$Z\mapsto Z^*$ when $p=q\geq 2$, $Z\mapsto\overline{Z}$
$\mathbb{H}$	$\mathrm{Sp}(p,q)$	$Z\mapsto Z^*$ when $p=q$

Proof. By the discussion in §3.2.4, it suffices to show that Γ normalises $\tau(G')$ and $\Gamma'=\mathrm{Ad}(\Gamma)$, where Γ' is the finite subgroup of $\mathrm{Aut}(\mathfrak{g})$ given in Table 2.10. Let $f:\mathcal{H}_{q,q}(\mathbb{K})\to\mathcal{H}_{q,q}(\mathbb{K})$ denote the map $Z\mapsto Z^*$, and let $c:\mathcal{H}_{p,q}(\mathbb{C})\to\mathcal{H}_{p,q}(\mathbb{C})$ denote the map $Z\mapsto\overline{Z}$. Given $L=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\in G'$, we find

$$f\circ\tau\begin{pmatrix}A&B\\C&D\end{pmatrix}\circ f^{-1}=\tau\begin{pmatrix}D&C\\B&A\end{pmatrix},\qquad c\circ\tau(L)\circ c^{-1}=\tau(\overline{L}).$$

Therefore, $\mathrm{Ad}(f):\mathfrak{g}\to\mathfrak{g}$ and $\mathrm{Ad}(c):\mathfrak{su}(p,q)\to\mathfrak{su}(p,q)$ are given by

$$\mathrm{Ad}(f)=\mathrm{Ad}\begin{pmatrix}0&\mathrm{I}_p\\\mathrm{I}_p&0\end{pmatrix},\qquad\mathrm{Ad}(c):X\mapsto\overline{X},$$

so $\Gamma'=\mathrm{Ad}(\Gamma)$, as desired. □

3.5 The case $\mathfrak{sp}(2n,\mathbb{R})$

In this section, we construct the symmetric space corresponding to $\mathfrak{sp}(2n,\mathbb{R})$, and compute its isometry group. Throughout this section, assume $n\geq 1$.

3.5.1 The symmetric space

In this subsection, we construct the *Seigel half-space* $\mathcal{SH}_n$, which is the irreducible symmetric space of non-compact type corresponding to the Lie algebra $\mathfrak{sp}(2n,\mathbb{R})$. The underlying smooth manifold of $\mathcal{SH}_n$ is the open subset of $\mathrm{Sym}(n,\mathbb{C})$ defined by

$$\mathcal{SH}_n:=\Big\{X+\mathbf{i}Y\in\mathrm{Sym}(n,\mathbb{C}):\ Y>0\Big\}.$$

Here, $Y>0$ means that Y is positive-definite.

Let $\omega:\mathbb{C}^{2n}\times\mathbb{C}^{2n}\to\mathbb{C}$ denote the skew-symmetric bilinear form given by

$$\omega(v,w):=v^\top Jw,\qquad\text{where } J=\begin{pmatrix}0&\mathrm{I}_n\\-\mathrm{I}_n&0\end{pmatrix},$$

and let $h:\mathbb{C}^{2n}\times\mathbb{C}^{2n}\to\mathbb{C}$ denote the Hermitian form given by

$$h(v,w):=v^*(-\mathbf{i}J)w.$$

Let $\mathrm{Gr}_n(\mathbb{C}^{2n})$ denote the Grassmannian of n-dimensional $\mathbb{C}$-subspaces of $\mathbb{C}^{2n}$, and let $F : \mathrm{M}_n(\mathbb{C}) \to \mathrm{Gr}_n(\mathbb{C}^{2n})$ denote the smooth embedding

$$Z \mapsto \Big\{(Zv, v) : v \in \mathbb{C}^n\Big\}.$$

From §3.4.1, recall that the image of F is the open subset

$$\mathcal{U} = \Big\{V \in \mathrm{Gr}_q(\mathbb{K}^n) : V \cap (\mathbb{K}^p \oplus 0) = 0\Big\}.$$

Proposition 3.28. *The image of $\mathcal{SH}_n$ under F is given by*

$$F(\mathcal{SH}_n) = \Big\{V \in \mathrm{Gr}_n(\mathbb{C}^{2n}) : \omega|_{V\times V} \equiv 0,\ h|_{V\times V} < 0\Big\}.$$

Here, $\omega|_{V\times V} \equiv 0$ means that $\omega(v, w) = 0$ for all $v, w \in V$, and $h|_{V\times V} < 0$ means that the restriction of h to V is negative-definite.

Proof of Proposition 3.28. First, suppose $V \in \mathrm{Gr}_n(\mathbb{C}^{2n})$ and $h|_{V\times V} < 0$. Let us show that V belongs to $\mathcal{U}$. Indeed, if $v \in \mathbb{C}^n$, then $h((v, 0), (v, 0)) = 0$, so $v = 0$. Therefore, $V \cap (\mathbb{C}^n \oplus 0) = 0$. Next, given $Z \in \mathrm{M}_n(\mathbb{C})$, the following statements are equivalent:

(i) Z is symmetric.

(ii) $\omega((Zv, v), (Zv, v)) = v^\top(Z^\top - Z)w$ is zero for all $v, w \in \mathbb{C}^n$.

(iii) $\omega|_{F(Z)\times F(Z)} \equiv 0$.

A similar argument shows that $\mathrm{Im}(Z) > 0$ if and only if h is negative-definite on $F(Z)$. □

The Lie group $\mathrm{Sp}(2n, \mathbb{R})$ acts smoothly on $\mathrm{Gr}_n(\mathbb{C}^{2n})$ by $A\cdot V = A(V)$. Since $\mathrm{Sp}(2n, \mathbb{R})$ preserves the forms ω and h, this action restricts to an action on $F(\mathcal{SH}_n)$. We equip $\mathcal{SH}_n$ with the smooth action of $\mathrm{Sp}(2n, \mathbb{R})$ induced by the diffeomorphism $F|_{\mathcal{SH}_n} : \mathcal{SH}_n \cong F(\mathcal{SH}_n)$.

Proposition 3.29. *The action of $\mathrm{Sp}(2n, \mathbb{R})$ on $\mathcal{SH}_n$ is given by*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ + B)(CZ + D)^{-1}.$$

Moreover, this action is transitive, and the isotropy subgroup at $\mathbf{i}\,\mathrm{I}_n$ is

$$\Phi(\mathrm{U}(n)) = \left\{\begin{pmatrix} A & B \\ -B & A \end{pmatrix} : A + \mathbf{i}B \in \mathrm{U}(n)\right\},$$

where Φ is defined in §2.1.4.

Proof. The proof of the first sentence is the same as the proof given in Proposition 3.23. To see that the action is transitive, fix $X + \mathbf{i}Y \in \mathcal{SH}_n$, and observe that the following matrices belong to $\mathcal{SH}_n$:

$$\begin{pmatrix} \mathrm{I}_n & X \\ 0 & \mathrm{I}_n \end{pmatrix}, \qquad \begin{pmatrix} \sqrt{Y} & 0 \\ 0 & \sqrt{Y^{-1}} \end{pmatrix},$$

where $\sqrt{Y}$ denotes the unique positive-definite matrix such that $\sqrt{Y}\sqrt{Y} = Y$. We find

$$\begin{pmatrix} \mathrm{I}_n & X \\ 0 & \mathrm{I}_n \end{pmatrix}\begin{pmatrix} \sqrt{Y} & 0 \\ 0 & \sqrt{Y^{-1}} \end{pmatrix} \cdot \mathbf{i}\,\mathrm{I}_n = X + \mathbf{i}Y.$$

It is easy to compute the isotropy subgroup at $\mathbf{i}\,\mathrm{I}_n$. □

Proposition 3.30. *On $T_{\mathbf{i}\,\mathrm{I}_n}\mathcal{SH}_n \cong \mathrm{Sym}(n,\mathbb{C})$, define $\langle V, W\rangle := \mathrm{Re}\,\mathrm{tr}(V\overline{W})$. Then*

(i) *$\langle\cdot,\cdot\rangle$ is a real inner product which is $\Phi(\mathrm{U}(n))$-invariant with respect to the isotropy representation. Here, Φ denotes the embedding given in §2.1.4.*

(ii) *Let g denote the unique $\mathrm{Sp}(2n,\mathbb{R})$-invariant metric on $\mathcal{SH}_n$ such that $g_{\mathbf{i}\,\mathrm{I}_n} = \langle\cdot,\cdot\rangle$. Then*

$$g_{X+\mathbf{i}Y}(V,W) = \mathrm{Re}\,\mathrm{tr}(Y^{-1}VY^{-1}\overline{W})$$

for all $X+\mathbf{i}Y$ in $\mathcal{SH}_n$ and $V, W \in T_{X+\mathbf{i}Y}\mathcal{SH}_n \cong \mathrm{Sym}(n,\mathbb{C})$.

Proof. For $L = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)$ in $\mathrm{Sp}(2n,\mathbb{R})$, let $\tau(L) : \mathcal{SH}_n \to \mathcal{SH}_n$ denote the diffeomorphism given by $Z \mapsto L\cdot Z$. The differential of $\tau(L)$ at Z is given by

$$d\tau(L)_Z : V \mapsto (A-(L\cdot Z)C)V(CZ+D)^{-1}.$$

Let us show (i). It is easy to see that $\langle\cdot,\cdot\rangle$ is a real inner product. Suppose $L = \left(\begin{smallmatrix} A & B \\ -B & A\end{smallmatrix}\right)$ belongs to $\Phi(\mathrm{U}(n))$, and fix $V, W \in T_{\mathbf{i}\,\mathrm{I}_n}\mathcal{SH}_n$. Then

$$\begin{aligned}\langle d\tau(L)_{\mathbf{i}\,\mathrm{I}_n}V, d\tau(L)_{\mathbf{i}\,\mathrm{I}_n}W\rangle &= \mathrm{Re}\,\mathrm{tr}\left((A+\mathbf{i}B)V(A-\mathbf{i}B)^{-1}(A-\mathbf{i}B)\overline{W}(A+\mathbf{i}B)^{-1}\right)\\ &= \mathrm{Re}\,\mathrm{tr}(V\overline{W}).\end{aligned}$$

To show (ii), fix $Z = X+\mathbf{i}Y \in \mathcal{SH}_n$. Let $L = \left(\begin{smallmatrix} \mathrm{I}_n & X \\ 0 & \mathrm{I}_n\end{smallmatrix}\right)\left(\begin{smallmatrix} \sqrt{Y} & 0 \\ 0 & \sqrt{Y^{-1}}\end{smallmatrix}\right)$. Then $L^{-1}\cdot Z = \mathbf{i}\,\mathrm{I}_n$, and $d(L^{-1})_Z V = Y^{-1}V$. Therefore,

$$g_Z(V,W) = g_{\mathbf{i}\,\mathrm{I}_n}\left(d(L^{-1})_Z V, d(L^{-1})_Z W\right) = \mathrm{Re}\,\mathrm{tr}(Y^{-1}VY^{-1}\overline{W}),$$

as desired. ☐

Henceforth, we equip $\mathcal{SH}_n$ with the Riemannian metric g given in Proposition 3.30. The Riemannian manifold $\mathcal{SH}_n$ is called the *Siegel half-space*. The following Proposition is an immediate consequence of Proposition 3.29 and Algorithm 3.10:

Proposition 3.31. *The Siegel half-space $\mathcal{SH}_n$ is the irreducible symmetric space of non-compact type corresponding to the Lie algebra $\mathfrak{sp}(2n,\mathbb{R})$.*

3.5.2 The isometry group

Proposition 3.32. *The isometry group of $\mathcal{SH}_n$ is generated by the action of $\mathrm{Sp}(2n,\mathbb{R})$ given in Proposition 3.29, and the map $f : X+\mathbf{i}Y \mapsto -X+\mathbf{i}Y$.*

Proof. By the discussion in §3.2.4, it suffices to show that

(i) f normalises $\tau(\mathrm{Sp}(2n,\mathbb{R}))$, where $\tau : \mathrm{Sp}(2n,\mathbb{R}) \to \mathrm{Isom}(\mathcal{SH}_n)$ is the action of $\mathrm{Sp}(2n,\mathbb{R})$ on $\mathcal{SH}_n$.

(ii) The image of f under the adjoint representation $\mathrm{Ad} : \mathrm{Isom}(\mathcal{SH}_n) \to \mathrm{Aut}(\mathfrak{sp}(2n,\mathbb{R}))$ is given by

$$\mathrm{Ad}(f) : \mathfrak{sp}(2n,\mathbb{R}) \to \mathfrak{sp}(2n,\mathbb{R}), \qquad \begin{pmatrix} X & Y \\ Z & -X^\top\end{pmatrix} \mapsto \begin{pmatrix} X & -Y \\ -Z & -X^\top\end{pmatrix}.$$

Indeed, given $\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in \mathrm{Sp}(2n,\mathbb{R})$, we find

$$f\circ\tau\begin{pmatrix} A & B \\ C & D\end{pmatrix}\circ f^{-1} = \tau\begin{pmatrix} A & -B \\ -C & D\end{pmatrix},$$

so the result follows. ☐

3.6 The case $\mathfrak{sp}(2n,\mathbb{C})$

In this section, we construct the symmetric space corresponding to $\mathfrak{sp}(2n,\mathbb{C})$, and compute its isometry group. Throughout this section, assume $n \geq 1$.

3.6.1 The symmetric space

In this subsection, we construct $\mathcal{P}(2n,\mathbb{C})_{\mathrm{Sp}}$, the irreducible symmetric space of non-compact type corresponding to the Lie algebra $\mathfrak{sp}(2n,\mathbb{C})$.

Recall that $\mathcal{P}(2n,\mathbb{C})$ denotes the set of positive-definite Hermitian $2n$ by $2n$ complex matrices. We define $\mathcal{P}(2n,\mathbb{C})_{\mathrm{Sp}}$ to be the embedded submanifold of $\mathcal{P}(2n,\mathbb{C})$ given by

$$\mathcal{P}(2n,\mathbb{C})_{\mathrm{Sp}} := \mathcal{P}(2n,\mathbb{C}) \cap \mathrm{Sp}(2n,\mathbb{C}).$$

Recall that the Lie group $\mathrm{GL}(2n,\mathbb{C})$ acts smoothly on $\mathcal{P}(2n,\mathbb{C})$ via $A \cdot P = APA^*$. Thus, $\mathrm{Sp}(2n,\mathbb{C})$ also acts smoothly on $\mathcal{P}(2n,\mathbb{C})$ via the same formula.

Proposition 3.33. *Consider the action of* $\mathrm{Sp}(2n,\mathbb{C})$ *on* $\mathcal{P}(2n,\mathbb{C})$ *given by* $A \cdot P = APA^*$.

(i) *The orbit of* I_n *is* $\mathcal{P}(2n,\mathbb{C})_{\mathrm{Sp}} = \mathcal{P}(2n,\mathbb{C}) \cap \mathrm{Sp}(2n,\mathbb{C})$.

(ii) *The isotropy subgroup at* I_n *is* $\Psi(\mathrm{Sp}(n)) = \mathrm{Sp}(2n,\mathbb{C}) \cap \mathrm{U}(2n)$, *where* Ψ *is the embedding defined in §2.1.4.*

Proof. The only non-trivial part of the proof is to show that every element of $\mathcal{P}(2n,\mathbb{C})_{\mathrm{Sp}}$ belongs to the orbit of I_n. Recall that the matrix exponential restricted to Hermitian matrices is bijective: $\exp : \mathrm{Herm}(2n,\mathbb{C}) \to \mathcal{P}(2n,\mathbb{C})$. Now, fix $P \in \mathcal{P}(2n,\mathbb{C})_{\mathrm{Sp}}$. Then $P^\top JP = J$. Let $X \in \mathrm{Herm}(2n,\mathbb{C})$ be the unique Hermitian matrix such that $P = \exp(X)$. Let us show that X belongs to the Lie algebra $\mathfrak{sp}(2n,\mathbb{C})$. It suffices to show that $X^\top J + JX = 0$. Indeed, the matrix $-J^{-1}X^\top J$ is also Hermitian, and

$$\exp(-J^{-1}X^\top J) = J^{-1}\left(\exp(X)^{-1}\right)^\top J = J^{-1}\left(P^{-1}\right)^\top J = P,$$

so injectivity of $\exp : \mathrm{Herm}(2n,\mathbb{C}) \to \mathcal{P}(2n,\mathbb{C})$ implies that $X = -J^{-1}X^\top J$. In particular, $\frac{1}{2}X$ belongs to $\mathfrak{sp}(2n,\mathbb{C})$, so $\exp(\frac{1}{2}X) \in \mathrm{Sp}(2n,\mathbb{C})$, and $\exp(\frac{1}{2}X) \cdot \mathrm{I}_n = P$. □

We equip $\mathcal{P}(2n,\mathbb{C})_{\mathrm{Sp}}$ with the Riemannian metric g induced by the Riemannian metric on $\mathcal{P}(2n,\mathbb{C})$ given in Proposition 3.13. Thus,

$$g_P(V,W) = \operatorname{Re}\operatorname{tr}(P^{-1}VP^{-1}W)$$

for all $P \in \mathcal{P}(2n,\mathbb{C})_{\mathrm{Sp}}$ and $V, W \in T_P\mathcal{P}(2n,\mathbb{C})_{\mathrm{Sp}} \subseteq \mathrm{Herm}(2n,\mathbb{C})$. Since $\mathrm{Sp}(2n,\mathbb{C}) \subseteq \mathrm{GL}(2n,\mathbb{C})$, and the metric on $\mathcal{P}(2n,\mathbb{C})$ is $\mathrm{GL}(2n,\mathbb{C})$-invariant, it follows that g is $\mathrm{Sp}(2n,\mathbb{C})$-invariant. Thanks to Algorithm 3.10, we have shown the following:

Proposition 3.34. *The Riemannian manifold* $\mathcal{P}(2n,\mathbb{C})_{\mathrm{Sp}}$ *is the irreducible symmetric space of non-compact type corresponding to the Lie algebra* $\mathfrak{sp}(2n,\mathbb{C})$.

3.6.2 The isometry group

Proposition 3.35. *The isometry group of* $\mathcal{P}(2n,\mathbb{C})_{\mathrm{Sp}}$ *is generated by the action of* $\mathrm{Sp}(2n,\mathbb{C})$ *and* $f : X \mapsto \overline{X}$.

Proof. By the discussion in §3.2.4, it suffices to show that

(i) f normalises $\tau(\mathrm{Sp}(2n,\mathbb{C}))$, where $\tau : \mathrm{Sp}(2n,\mathbb{C}) \to \mathrm{Isom}(\mathcal{P}(2n,\mathbb{C})_{\mathrm{Sp}})$ is the action of $\mathrm{Sp}(2n,\mathbb{R})$ on $\mathcal{P}(2n,\mathbb{C})_{\mathrm{Sp}}$.

(ii) The image of f under the adjoint representation $\mathrm{Ad} : \mathrm{Isom}(\mathcal{P}(2n,\mathbb{C})_{\mathrm{Sp}}) \to \mathrm{Aut}(\mathfrak{sp}(2n,\mathbb{C}))$ is the automorphism of $\mathfrak{sp}(2n,\mathbb{C})$ given by $X \mapsto \overline{X}$.

Indeed, given $A \in \mathrm{Sp}(2n,\mathbb{C})$, we find $f \circ \tau(A) \circ f^{-1} = \tau(\overline{A})$. The result follows. □

3.7 The case $\mathfrak{so}(n,\mathbb{C})$

In this section, we construct the symmetric space corresponding to $\mathfrak{so}(n,\mathbb{C})$, and compute its isometry group. Throughout this section, assume $n \geq 3$.

3.7.1 The symmetric space

In this subsection, we construct $\mathcal{B}_{\mathrm{Skew}(n,\mathbb{R})}$, the irreducible symmetric space of non-compact type corresponding to the Lie algebra $\mathfrak{so}(n,\mathbb{C})$. The underlying smooth manifold of $\mathcal{B}_{\mathrm{Skew}(n,\mathbb{R})}$ is the open subset of the vector space $\mathrm{Skew}(n,\mathbb{R})$ given by

$$\begin{aligned}\mathcal{B}_{\mathrm{Skew}(n,\mathbb{R})} &:= \Big\{X \in \mathrm{Skew}(n,\mathbb{R}) : X^\top X - \mathrm{I}_n < 0\Big\}\\ &= \Big\{X \in \mathrm{Skew}(n,\mathbb{R}) : \|X\|_{\mathrm{op}} < 1\Big\}.\end{aligned}$$

Here, $X^\top X - \mathrm{I}_n < 0$ means that the symmetric matrix $X^\top X - \mathrm{I}_n$ is negative-definite, and $\|\cdot\|_{\mathrm{op}}$ is the operator norm on $\mathrm{M}_n(\mathbb{R})$ (see §3.4.2). Thus, $\mathcal{B}_{\mathrm{Skew}(n,\mathbb{R})}$ is precisely the open unit ball in the normed space $(\mathrm{Skew}(n,\mathbb{R}), \|\cdot\|_{\mathrm{op}})$.

Let $k, h : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ denote the symmetric bilinear forms given by

$$k(v,w) := v^\top \begin{pmatrix} 0 & \mathrm{I}_n \\ \mathrm{I}_n & 0 \end{pmatrix} w, \qquad h(v,w) := v^\top \begin{pmatrix} \mathrm{I}_n & 0 \\ 0 & -\mathrm{I}_n \end{pmatrix} w.$$

Let $\mathrm{Gr}_n(\mathbb{R}^{2n})$ denote the Grassmannian of n-dimensional real subspaces of $\mathbb{R}^{2n}$, and let $F : \mathrm{M}_n(\mathbb{R}) \to \mathrm{Gr}_n(\mathbb{R}^{2n})$ denote the smooth embedding

$$F(X) := \Big\{(Xv, v) : v \in \mathbb{R}^n\Big\}.$$

Proposition 3.36. *The image of $\mathcal{B}_{\mathrm{Skew}(n,\mathbb{R})}$ under F is*

$$F(\mathcal{B}_{\mathrm{Skew}(n,\mathbb{R})}) = \Big\{V \in \mathrm{Gr}_n(\mathbb{R}^{2n}) : k|_{V\times V} \equiv 0,\ h|_{V\times V} < 0\Big\}.$$

Here, $k|_{V\times V} \equiv 0$ means that $k(v,w) = 0$ for all $v, w \in V$, and $h|_{V\times V} < 0$ means that the restriction of h to V is negative-definite.

Proof of 3.36. In the proof of Proposition 3.21, we have already shown that the set on the right-hand side is contained in the image of F, and that $X^\top X - \mathrm{I}_n < 0$ is equivalent to $h|_{F(X)\times F(X)} < 0$. Now, fix $X \in \mathrm{M}_n(\mathbb{R})$. Then the following are equivalent:

(i) X belongs to $\mathrm{Skew}(n,\mathbb{R})$.

(ii) $k((Xv,v),(Xw,w)) = v^\top(X^\top + X)w$ is zero for all $v, w \in \mathbb{R}^n$.

(iii) $k|_{F(X)\times F(X)} \equiv 0$.

This completes the proof. $\square$

The Lie group $\mathrm{O}(n,\mathbb{C})$ acts on the Grassmannian $\mathrm{Gr}_n(\mathbb{R}^{2n})$ by $L\cdot V := \Phi(L)(V)$, where $\Phi : \mathrm{M}_n(\mathbb{C}) \to \mathrm{M}_{2n}(\mathbb{R})$ is the embedding $A + \mathbf{i}C \mapsto \left(\begin{smallmatrix} A & -C \\ C & A \end{smallmatrix}\right)$ (see §2.1.4).

Proposition 3.37. *Consider the action of* $\mathrm{O}(n,\mathbb{C})$ *on* $\mathrm{Gr}_n(\mathbb{R}^{2n})$ *given by* $L\cdot V := \Phi(L)(V)$*.*

(i) *The orbit of* $0 \oplus \mathbb{R}^n$ *is* $F(\mathcal{B}_{\mathrm{Skew}(n,\mathbb{R})})$*.*

(ii) *The isotropy subgroup at* $0 \oplus \mathbb{R}^n$ *is* $\mathrm{O}(n)$*.*

Proof. Let $\rho : \mathbb{C}^n \to \mathbb{R}^{2n}$ denote the $\mathbb{R}$-linear isomorphism given by $v + \mathbf{i}w \mapsto (v, w)$. Recall that $\rho(Lv) = \Phi(L)\rho(v)$ for $L \in \mathrm{M}_n(\mathbb{C})$ and $v \in \mathbb{C}^n$. Using ρ, let us identify $\mathrm{Gr}_n(\mathbb{R}^{2n})$ with the real Grassmannian $\mathrm{Gr}_n^{\mathbb{R}}(\mathbb{C}^n)$ of n-dimensional real subspaces of $\mathbb{C}^n$ considered as a real vector space. Under this identification, the action of $\mathrm{O}(n,\mathbb{C})$ on $\mathrm{Gr}_n^{\mathbb{R}}(\mathbb{C}^n)$ is $L \cdot V = L(V)$, and $0 \oplus \mathbb{R}^n \leftrightarrow \mathbf{i}\mathbb{R}^n$. Let $b : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ denote the standard bilinear form on $\mathbb{C}^n$ given by $b(v, w) := v^\top w$. Then

$$b(v, w) = h(\rho(v), \rho(w)) + \mathbf{i}k(\rho(v), \rho(w)).$$

It follows that, under the identification $\mathrm{Gr}_n(\mathbb{R}^{2n}) \cong \mathrm{Gr}_n^{\mathbb{R}}(\mathbb{C}^n)$, we can write

$$F(\mathcal{B}_{\mathrm{Skew}(n,\mathbb{R})}) = \Big\{ V \in \mathrm{Gr}_n^{\mathbb{R}}(\mathbb{C}^n) : b|_{V\times V} \text{ is real-valued and negative-definite} \Big\}.$$

Let us show (i). Observe that the restriction of b to $\mathbf{i}\mathbb{R}^n$ is real-valued and negative definite. Since $\mathrm{O}(n,\mathbb{C})$ preserves b by definition, it follows that the orbit of $\mathbf{i}\mathbb{R}^n$ is contained in $F(\mathcal{B}_{\mathrm{Skew}(n,\mathbb{R})})$. Conversely, suppose $V \in \mathrm{Gr}_n^{\mathbb{R}}(\mathbb{C}^n)$, and $b|_{V\times V}$ is real-valued and negative-definite. Let $\mathbf{i}v_1, \ldots, \mathbf{i}v_n$ be an orthonormal basis for V with respect to the real inner product $-b$. Let A be the matrix whose columns are $v_1, \ldots, v_n$. Then A belongs to $\mathrm{O}(n,\mathbb{C})$, and $A \cdot \mathbf{i}\mathbb{R}^n = V$.

Next, let us show (ii). Clearly, if $A \in \mathrm{O}(n)$, then $A \cdot \mathbf{i}\mathbb{R}^n = \mathbf{i}\mathbb{R}^n$. Conversely, suppose $A \in \mathrm{O}(n,\mathbb{C})$ and $A \cdot \mathbf{i}\mathbb{R}^n = \mathbf{i}\mathbb{R}^n$. Then $Ae_i \in \mathbb{R}^n$ for $i = 1, \ldots, n$, so A is real. $\square$

We equip $\mathcal{B}_{\mathrm{Skew}(n,\mathbb{R})}$ with the action by $\mathrm{O}(n,\mathbb{C})$ induced via the diffeomorphism $F|_{\mathcal{B}_{\mathrm{Skew}(n,\mathbb{R})}} : \mathcal{B}_{\mathrm{Skew}(n,\mathbb{R})} \cong F(\mathcal{B}_{\mathrm{Skew}(n,\mathbb{R})})$.

Proposition 3.38. *The action of* $\mathrm{O}(n,\mathbb{C})$ *on* $\mathcal{B}_{\mathrm{Skew}(n,\mathbb{R})}$ *is given by*

$$(A + \mathbf{i}C) \cdot X = (AX - C)(CX + A)^{-1}.$$

Moreover, this action is transitive, and the isotropy subgroup at $0 \in \mathcal{B}_{\mathrm{Skew}(n,\mathbb{R})}$ *is* $\mathrm{O}(n)$*.*

Proof. The first sentence follows from Proposition 3.23. The second sentence follows from Proposition 3.38, since $F(0) = 0 \oplus \mathbb{R}^n$. $\square$

Observe that $\mathcal{B}_{\mathrm{Skew}(n,\mathbb{R})}$ is an embedded submanifold of the hyperbolic Grassmannian $\mathcal{H}_{n,n}(\mathbb{R})$ constructed in §3.4.2. We endow $\mathcal{B}_{\mathrm{Skew}(n,\mathbb{R})}$ with the Riemannian metric g induced from the Riemannian metric on $\mathcal{H}_{n,n}(\mathbb{R})$. Thus,

$$g_X(V, W) = \mathrm{tr}\Big((\mathrm{I}_n - XX^\top)^{-1} V (\mathrm{I}_n - X^\top X)^{-1} W^\top \Big)$$

for all $X \in \mathcal{B}_{\mathrm{Skew}(n,\mathbb{R})}$ and $V, W \in T_X\mathcal{B}_{\mathrm{Skew}(n,\mathbb{R})} \cong \mathrm{Skew}(n,\mathbb{R})$. Since $\Phi(\mathrm{O}(n,\mathbb{C})) \subseteq \mathrm{O}(n,n)$ (§2.2) and the metric on $\mathcal{H}_{n,n}(\mathbb{R})$ is $\mathrm{O}(n,n)$-invariant (§3.4), it follows that g is $\mathrm{O}(n,\mathbb{C})$-invariant. The action of the identity component $\mathrm{SO}(n,\mathbb{C})$ on $\mathcal{B}_{\mathrm{Skew}(n,\mathbb{R})}$ is also transitive (since $\mathcal{B}_{\mathrm{Skew}(n,\mathbb{R})}$ is connected), and the isotropy subgroup of this action is $\mathrm{SO}(n) = \mathrm{SO}(n,\mathbb{C}) \cap \mathrm{O}(n)$. Therefore, via Algorithm 3.10, we have shown the following:

Proposition 3.39. *The Riemannian manifold* $\mathcal{B}_{\mathrm{Skew}(n,\mathbb{R})}$ *is the irreducible symmetric space of non-compact type corresponding to the Lie algebra* $\mathfrak{so}(n,\mathbb{C})$*.*

3.7.2 Isometry group

Proposition 3.40. *The isometry group of* $\mathcal{B}_{\mathrm{Skew}(n,\mathbb{R})}$ *is generated by the action of* $\mathrm{O}(n,\mathbb{C})$ *given in Proposition 3.38 and* $f : X \mapsto X^\top$*.*

Proof. By the discussion in §3.2.4, it suffices to show that

(i) f normalises $\tau(\mathrm{O}(n,\mathbb{C}))$, where $\tau : \mathrm{O}(n,\mathbb{C}) \to \mathrm{Isom}(\mathcal{B}_{\mathrm{Skew}(n,\mathbb{R})})$ is the action of $\mathrm{O}(n,\mathbb{C})$ on $\mathcal{B}_{\mathrm{Skew}(n,\mathbb{R})}$.

(ii) The adjoint representation $\mathrm{Ad} : \mathrm{Isom}(\mathcal{B}_{\mathrm{Skew}(n,\mathbb{R})}) \to \mathrm{Aut}(\mathfrak{so}(n,\mathbb{C}))$ maps f to the automorphism of $\mathfrak{so}(n,\mathbb{C})$ given by $X \mapsto \overline{X}$.

Indeed, given $L \in \mathrm{O}(n,\mathbb{C})$, we find $f \circ \tau(L) \circ f^{-1} = \tau(\overline{L})$. The result follows. ☐

3.8 The case $\mathfrak{so}(n,\mathbb{H})$

In this section, we construct the symmetric space corresponding to $\mathfrak{so}(n,\mathbb{H})$, and compute its isometry group. Throughout this section, assume $n \geq 5$.

3.8.1 The symmetric space

In this subsection, we construct $\mathcal{B}_{\mathrm{Skew}(n,\mathbb{C})}$, the irreducible symmetric space of non-compact type corresponding to the Lie algebra $\mathfrak{so}(n,\mathbb{H})$. The underlying smooth manifold of $\mathcal{B}_{\mathrm{Skew}(n,\mathbb{C})}$ is the open subset of the vector space $\mathrm{Skew}(n,\mathbb{C})$ given by

$$\begin{aligned}\mathcal{B}_{\mathrm{Skew}(n,\mathbb{C})} &:= \Big\{Z \in \mathrm{Skew}(n,\mathbb{C}) : Z^*Z - \mathrm{I}_n < 0\Big\}\\ &= \Big\{Z \in \mathrm{Skew}(n,\mathbb{C}) : \|Z\|_{\mathrm{op}} < 1\Big\}.\end{aligned}$$

Here, $Z^*Z - \mathrm{I}_n < 0$ means that the Hermitian matrix $Z^*Z - \mathrm{I}_n$ is negative-definite, and $\|\cdot\|_{\mathrm{op}}$ is the operator norm on $\mathrm{M}_n(\mathbb{C})$ (see §3.4.2). Thus, $\mathcal{B}_{\mathrm{Skew}(n,\mathbb{C})}$ is precisely the open unit ball in the normed space $(\mathrm{Skew}(n,\mathbb{C}), \|\cdot\|_{\mathrm{op}})$.

Let $k, h : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \to \mathbb{C}$ denote the symmetric-bilinear and Hermitian forms given by

$$k(v,w) := v^\top \begin{pmatrix} 0 & \mathrm{I}_n \\ \mathrm{I}_n & 0 \end{pmatrix} w, \qquad h(v,w) := v^* \begin{pmatrix} \mathrm{I}_n & 0 \\ 0 & -\mathrm{I}_n \end{pmatrix} w.$$

Let $\mathrm{Gr}_n(\mathbb{C}^{2n})$ denote the Grassmannian of n-dimensional complex subspaces of $\mathbb{C}^{2n}$, and let $F : \mathrm{M}_n(\mathbb{C}) \to \mathrm{Gr}_n(\mathbb{C}^{2n})$ denote the smooth embedding

$$F(Z) := \Big\{(Zv, v) : v \in \mathbb{C}^n\Big\}.$$

Proposition 3.41. *The image of* $\mathcal{B}_{\mathrm{Skew}(n,\mathbb{C})}$ *under* F *is*

$$F(\mathcal{B}_{\mathrm{Skew}(n,\mathbb{C})}) = \Big\{V \in \mathrm{Gr}_n(\mathbb{C}^{2n}) : k|_{V\times V} \equiv 0,\ h|_{V\times V} < 0\Big\}.$$

Here, $k|_{V\times V}\equiv 0$ means that $k(v,w)=0$ for all $v,w\in V$, and $h|_{V\times V}<0$ means that the restriction of h to V is negative-definite. The proof of Proposition 3.41 is the same as the proof of Proposition 3.36.

The Lie group $\mathrm{SO}(n,\mathbb{H})$ acts on the Grassmannian $\mathrm{Gr}_n(\mathbb{C}^{2n})$ by $L\cdot V:=\Psi(L)(V)$, where $\Psi:\mathrm{M}_n(\mathbb{H})\to\mathrm{M}_{2n}(\mathbb{C})$ is the embedding $A+\mathbf{j}C\mapsto\left(\begin{smallmatrix}A & -\overline{C}\\ C & \overline{A}\end{smallmatrix}\right)$ (see §2.1.4).

Proposition 3.42. *Consider the action of* $\mathrm{SO}(n,\mathbb{H})$ *on* $\mathrm{Gr}_n(\mathbb{C}^{2n})$*. Then*

(i) *The orbit of* $0\oplus\mathbb{C}^n$ *is* $F(\mathcal{B}_{\mathrm{Skew}(n,\mathbb{C})})$*.*

(ii) *The isotropy subgroup at* $0\oplus\mathbb{C}^n$ *is* $\mathrm{U}(n)$*.*

Proof. We consider $\mathbb{H}^n$ as a $\mathbb{C}$-vector space, where scalars act from the right. Recall that $\rho:\mathbb{H}^n\to\mathbb{C}^{2n}$ given by $v+\mathbf{j}w\mapsto(v,w)$ is a $\mathbb{C}$-linear isomorphism. We have $\rho(Lv)=\Psi(L)\rho(v)$ for $L\in\mathrm{M}_n(\mathbb{H})$ and $v\in\mathbb{C}^n$. Using ρ, let us identify $\mathrm{Gr}(\mathbb{C}^{2n})$ with the complex Grassmannian $\mathrm{Gr}_n^{\mathbb{C}}(\mathbb{H}^n)$ of n-dimensional complex subspaces of $\mathbb{H}^n$ considered as a complex vector space. Under this identification, the action of $\mathrm{SO}(n,\mathbb{H})$ on $\mathrm{Gr}_n^{\mathbb{C}}(\mathbb{H}^n)$ is given by $L\cdot V=L(V)$, and $0\oplus\mathbb{C}^n\leftrightarrow\mathbf{j}\mathbb{C}^n$. Let $c:\mathbb{H}^n\times\mathbb{H}^n\to\mathbb{H}$ denote the skew-Hermitian form on $\mathbb{H}^n$ given by $c(v,w):=v^*\mathbf{i}w$. Then

$$c(v,w)=\mathbf{i}h(\rho(v),\rho(w))+\mathbf{k}k(\rho(v),\rho(w)).$$

It follows that, under the identification $\mathrm{Gr}_n(\mathbb{C}^{2n})\cong\mathrm{Gr}_n^{\mathbb{C}}(\mathbb{H}^n)$, we can write

$$F(\mathcal{B}_{\mathrm{Skew}(n,\mathbb{C})})=\left\{V\in\mathrm{Gr}_n^{\mathbb{C}}(\mathbb{H}^n):\mathbf{i}c|_{V\times V}\text{ is complex-valued, positive-definite Hermitian}\right\}.$$

Let us show (i). Observe that the restriction of $\mathbf{i}c$ to $\mathbf{j}\mathbb{C}^n$ is a complex-valued positive-definite Hermitian inner product. Since $\mathrm{SO}(n,\mathbb{H})$ preserves c by definition, it follows that the orbit of $\mathbf{j}\mathbb{C}^n$ is contained in $F(\mathcal{B}_{\mathrm{Skew}(n,\mathbb{C})})$. Conversely, suppose $V\in\mathrm{Gr}_n^{\mathbb{C}}(\mathbb{H}^n)$, and $\mathbf{i}c|_{V\times V}$ is complex-valued and a positive-definite Hermitian inner product. Let $v_1\mathbf{j},\dots,v_n\mathbf{j}$ be an orthonormal basis for V. Let L be the matrix whose columns are $v_1,\dots,v_n$. Then L belongs to $\mathrm{SO}(n,\mathbb{H})$, and $L\cdot\mathbf{j}\mathbb{C}^n=V$.

Next, let us show (ii). Suppose $A\in\mathrm{U}(n)$. Then $A\cdot\mathbf{j}\mathbb{C}^n=\mathbf{j}\overline{A}\mathbb{C}^n=\mathbf{j}\mathbb{C}^n$. Conversely, suppose $L\in\mathrm{SO}(n,\mathbb{H})$, and $L\cdot\mathbf{j}\mathbb{C}^n=\mathbf{j}\mathbb{C}^n$. Then $Le_i\in\mathbb{C}^n$, so L is complex. ☐

Equip $\mathcal{B}_{\mathrm{Skew}(n,\mathbb{C})}$ with the action of $\mathrm{SO}(n,\mathbb{H})$ induced by the diffeomorphism $F|_{\mathcal{B}_{\mathrm{Skew}(n,\mathbb{C})}}:\mathcal{B}_{\mathrm{Skew}(n,\mathbb{C})}\cong F(\mathcal{B}_{\mathrm{Skew}(n,\mathbb{C})})$.

Proposition 3.43. *The action of* $\mathrm{SO}(n,\mathbb{H})$ *on* $\mathcal{B}_{\mathrm{Skew}(n,\mathbb{C})}$ *is given by*

$$(A+\mathbf{j}C)\cdot Z=(AZ-\overline{C})(CZ+\overline{A})^{-1}.$$

Moreover, this action is transitive, and the isotropy subgroup at $0\in\mathcal{B}_{\mathrm{Skew}(n,\mathbb{C})}$ *is* $\mathrm{U}(n)$*.*

Proof. The first sentence follows from the Proposition 3.23, and the second sentence follows from Proposition 3.43, since $F(0)=0\oplus\mathbb{C}^n$. ☐

Observe that $\mathcal{B}_{\mathrm{Skew}(n,\mathbb{C})}$ is an embedded submanifold of the hyperbolic Grassmannian $\mathcal{H}_{n\times n}(\mathbb{C})$, which is the symmetric space corresponding to the Lie algebra $\mathfrak{su}(n,n)$ (see §3.4.2). We endow $\mathcal{B}_{\mathrm{Skew}(n,\mathbb{C})}$ with the Riemannian metric g induced from the Riemannian metric on $\mathcal{H}_{n,n}(\mathbb{C})$. Thus,

$$g_Z(V,W)=\operatorname{Re}\operatorname{tr}\left((\mathrm{I}_n-ZZ^*)^{-1}V(\mathrm{I}_n-Z^*Z)^{-1}W^*\right)$$

for all $Z \in \mathcal{B}_{\mathrm{Skew}(n,\mathbb{C})}$ and $V, W \in T_Z\mathcal{B}_{\mathrm{Skew}(n,\mathbb{C})} \cong \mathrm{Skew}(n,\mathbb{C})$. Since $\Psi(\mathrm{SO}(n,\mathbb{H})) \subseteq \mathrm{SU}(n,n)$ (§2.2), and the metric on $\mathcal{H}_{n,n}(\mathbb{C})$ is $\mathrm{SU}(n,n)$-invariant (§3.4), it follows that g is $\mathrm{SO}(n,\mathbb{H})$-invariant. Therefore, thanks to Algorithm 3.10, we have shown the following:

Proposition 3.44. *The Riemannian manifold $\mathcal{B}_{\mathrm{Skew}(n,\mathbb{C})}$ is the irreducible symmetric space of non-compact type corresponding to the Lie algebra $\mathfrak{so}(n,\mathbb{H})$.*

3.8.2 Isometry group

Proposition 3.45. *The isometry group of $\mathcal{B}_{\mathrm{Skew}(n,\mathbb{C})}$ is generated by the action of $\mathrm{SO}(n,\mathbb{H})$ given in Proposition 3.43 and $f : X \mapsto \overline{X}$.*

Proof. By the discussion in §3.2.4, it suffices to show that

(i) f normalises $\tau(\mathrm{SO}(n,\mathbb{H}))$, where $\tau : \mathrm{SO}(n,\mathbb{H}) \to \mathrm{Isom}(\mathcal{B}_{\mathrm{Skew}(n,\mathbb{C})})$ is the action of $\mathrm{SO}(n,\mathbb{H})$ on $\mathcal{B}_{\mathrm{Skew}(n,\mathbb{C})}$.

(ii) The adjoint representation $\mathrm{Ad} : \mathrm{Isom}(\mathcal{B}_{\mathrm{Skew}(n,\mathbb{C})}) \to \mathrm{Aut}(\mathfrak{so}(n,\mathbb{H}))$ maps f to the automorphism of $\mathfrak{so}(n,\mathbb{H})$ given by $X + \mathbf{j}Y \mapsto \overline{X} + \mathbf{j}\overline{Y}$.

Indeed, given $L \in \mathrm{SO}(n,\mathbb{H})$, we find $f \circ \tau(A+\mathbf{j}C) \circ f^{-1} = \tau(\overline{A}+\mathbf{j}\overline{C})$. The result follows. □

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